Integrals Theorems Summary

For all the theorems assume the function, transformation, and vector fields contain functions that have continuous first partial derivatives.

Change of Variable for Double Integrals:
Let $x = g(u, v)$ and $y = h(u, v)$ be a continuous, one-to-one transformation. If the region S in the uv-plane is mapped to the region R in the xy-plane by the transformation then

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv,$$

where $\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|$ is the absolute value of the Jacobian. There is a very similar result for triple integrals.

Conservative Vector Fields:
Let $\mathbf{F}$ be a vector field that is defined on all $\mathbf{R}^3$ (or $\mathbf{R}^2$).
If $\operatorname{curl} \mathbf{F} = 0$, then $\mathbf{F}$ is conservative. In $\mathbf{R}^2$, the condition simplifies to $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.
In which case, $\mathbf{F}(x, y, z) = \nabla f(x, y, z)$ for some *potential* function $f(x, y, z)$ (which we can find by integrating in steps). If C is a curve that starts at the point $A(x_0, y_0, z_0)$ and ends at the point $B(x_1, y_1, z_1)$, then

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A).}$$

There are several consequences this fact including:

1. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent. Any curve from A to B will give the same value (If $\mathbf{F}$ is a conservative force field, then this is the Law of Conservation of Energy).
2. On any closed curve C, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

Green's Theorem:
Let $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ be a vector field that is defined on $\mathbf{R}^2$. If C is a positively oriented, simple closed curve that encloses the region D, then

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.}$$

In section 16.5, we discussed the so-called vector forms of Green's theorem which were (measuring the tangential and normal components of $\mathbf{F}$ along C respectively):

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\, dA, \quad \text{and} \quad \int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C -Q\, dx + P\, dy = \iint_D \operatorname{div} \mathbf{F}\, dA.$$

Stokes's Theorem:
Let $\mathbf{F}$ be a vector field that is defined on $\mathbf{R}^3$. If S is a oriented surface that is bounded by a simple, closed curve C with positive orientation, then

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.}$$

Divergence's (or Guass') Theorem:
Let $\mathbf{F}$ be a vector field that is defined on $\mathbf{R}^3$. If E is a simple solid region bounded by the surface S with positive (outward) orientation, then

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV.}$$