Conservation of Energy Discussion (from 16.3)

Here is a brief discussion of the origin of the term *conservative* for a vector field, $\mathbf{F}$, that is the gradient of some *potential* function, f. Mathematically, this relationship is $\mathbf{F} = \nabla f$, but let's see where the terms come from.

FIRST, let $\mathbf{F}(x, y, z)$ be a force vector field that moves a particle of mass, m, from point A to point B along the curve C given by $r(t)$ with $a \le t \le b$ (remember we called such a path a flowline). By Newton's Second Law of Motion we get

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t) \qquad \text{(Force = mass x acceleration).}$$

Then the work done by the force on the particle is

$$\begin{aligned}
W = \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b m\mathbf{r}''(t) \cdot \mathbf{r}'(t)dt && \text{(by definition and the fact above)} \\
&= \frac{m}{2} \int_a^b \frac{d}{dt}[\mathbf{r}'(t) \cdot \mathbf{r}'(t)]dt && \text{(by the dot product derivative rule)} \\
&= \frac{m}{2} \int_a^b \frac{d}{dt}|\mathbf{r}'(t)|^2 dt && \text{(by the definition of magnitude)} \\
&= \frac{m}{2}|\mathbf{r}'(b)|^2 - \frac{m}{2}|\mathbf{r}'(a)|^2 && \text{(by the fundamental theorem of calculus)} \\
&= \frac{m}{2}|\mathbf{v}(b)|^2 - \frac{m}{2}|\mathbf{v}(a)|^2 && (\mathbf{v}(t) = \mathbf{r}'(t) = \text{ velocity })
\end{aligned}$$

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In physics, $K(t) = \frac{1}{2}m|\mathbf{v}(t)|^2$ is called the *kinetic energy* of an object. Thus, for ANY force vector field *along a flow line*, we have

$$\text{WORK } = K(B) - K(A) = \text{ the change in kinetic energy.}$$

SECOND, now let $\mathbf{F}$ also satisfy our mathematical definition of conservative, that is, $\mathbf{F} = \nabla f(x, y, z)$. In physics, the *potential energy* of the particle is defined to be $P(x, y, z) = -f(x, y, z)$, which means $\mathbf{F} = -\nabla P$. In which case, from the fundamental theorem of 16.3, we get

$$\begin{aligned}
W = \int_C \mathbf{F} \cdot d\mathbf{r} &= -\int_a^b \nabla P \cdot d\mathbf{r} && \text{(from above)} \\
&= -[P(\mathbf{r}(b)) - P(\mathbf{r}(a)) && \text{(by the fund. thm. of line integrals)]} \\
&= P(A) - P(B) \\
&= \text{negative change in potential energy}
\end{aligned}$$

Hence, IF $\mathbf{F} = \nabla f$, then we can say that the

$$\text{change in kinetic energy } = K(B) - K(A) = \text{ WORK } = P(A) - P(B) = \text{ change in potential energy}$$

Another way to say write this is:

$$P(B) + K(B) = P(A) + K(A).$$

So the sum of potential and kinetic energy is **conserved** from A to B in a conservative vector field. This is the law of conservation of energy.