16.4: Green's Theorem

Green's Theorem states: On a positively oriented, simple closed curve $C$ that encloses the region $D$ where $P$ and $Q$ have continuous partial derivatives, we have

$$\int_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA.$$ 

As noted in class, when working with positively oriented closed curve, $C$, we typically use the notation:

$$\oint_C P \, dx + Q \, dy = \int_C P \, dx + Q \, dy.$$ 

NOTES:

1. This theorem is for closed curves.

2. It is true for conservative and nonconservative vector fields. But for conservative vector fields the value of such an integral is just zero (remember that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ for a conservative vector field). So really this theorem is for nonconservative vector fields over closed curves.

3. This theorem gives an important relationship between the boundary a line integral of the boundary of a region and the double integral itself. These facts are useful in several ways:

   (a) Computing a line integral faster: This gives us options. If it is a pain to parameterize the closed curve, then we can instead do a double integral. Both ways work, but this theorem gives us options to choose a faster computation method.

   (b) Computing a double integral with a line integral: Sometimes it may be easier to work over the boundary than the interior. Green’s theorem gives us a connection between the two so that we can compute over the boundary. For example we found that we can find the area of a two-dimensional region in several way using line integrals as follows:

   $$\text{Area of } D = \iint_D 1 \, dA = \oint_C -y \, dx = \oint_C x \, dy = \frac{1}{2} \oint_C -y \, dx + x \, dy$$

4. We will interpret the physical significance of this result more in subsequent chapters. For now you need to be able to compute with it. The following page contains two examples.
• Compute \( \oint_C -2y^3 \, dx + 2x^3 \, dy \) where \( C \) is the circle of radius 3 centered at the origin.

ANSWER: Using Green’s theorem we need to describe the interior of the region in order to set up the bounds for our double integral. This is best described with polar coordinates, \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq r \leq 3 \). And we get

\[
\oint_C -2y^3 \, dx + 2x^3 \, dy = \iint_D (6x^2 + 6y^2) \, dA \\
= 6 \int_0^{2\pi} \int_0^3 r^2 r \, dr \, d\theta \\
= 6 \int_0^{2\pi} \left[ \frac{1}{4} r^4 \right]_0^3 \, d\theta \\
= 6 \int_0^{2\pi} \frac{81}{4} \, d\theta \\
= \frac{243}{2} \left[ \theta \right]_0^{2\pi} = 243\pi
\]

So if \( \mathbf{F}(x,y) = \langle -2y^3, 2x^3 \rangle \) was a force field say in units Newtons, then we just calculated

\[
\text{WORK} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C -2y^3 \, dx + 2x^3 \, dy = 243\pi \text{ Joules.}
\]

• Compute \( \oint_C x \, dx + xy^2 \, dy \) where \( C \) is the triangle with vertices (0,0), (2,0), (2,6).

ANSWER: Using Green’s theorem we need to describe the interior of the region in order to set up the bounds for our double integral. The triangle has sides with equations (in \( x \) and \( y \)) of \( y = 0 \), \( x = 2 \) and \( y = 3x \). If you graph the region, you see that it can be described as a ‘top/bottom’ region using \( 0 \leq x \leq 2 \) with \( 0 \leq y \leq 3x \). And we get

\[
\oint_C x \, dx + xy^2 \, dy = \iint_D (y^2 - 0) \, dA \\
= \int_0^2 \int_0^{3x} y^2 \, dy \, dx \\
= \int_0^2 \left[ \frac{1}{3} y^3 \right]_0^{3x} \, dx \\
= \int_0^2 9x^3 \, dx \\
= \frac{9}{4} x^4 \bigg|_0^2 = 36
\]

Remember if \( \mathbf{F}(x,y) = \langle x, xy^2 \rangle \) was a force field say in units Newtons, then we just calculated

\[
\text{WORK} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C x \, dx + xy^2 \, dy = 36 \text{ Joules.}
\]