Surface Integrals for Scalar Fields

\[ \int_S f(x, y, z) \, ds = \int_D f(\mathbf{r}(u, v)) \left| \mathbf{r}_u \times \mathbf{r}_v \right| \, da \]

Where \( S : z = f(x, y) \)

Let \( S \) be the boundary of the solid between \( z = 4 - x^2 - y^2 \) and the xy-plane.

**ANS**
- \( S_1 \) = the bottom \( z = 0 \) \( \Rightarrow \) \( x^2 + y^2 = 4 \)

**Parameterize:**
- \( x = r \cos(\theta) \)
- \( y = r \sin(\theta) \)
- \( z = 0 \)
- \( r \): \( 0 \leq r \leq 2 \)
- \( \theta \): \( 0 \leq \theta \leq \pi \)

\[
\left| \mathbf{r}_u \times \mathbf{r}_v \right| = \sqrt{\frac{d}{du} \frac{d}{dv} - \frac{d}{du} \frac{d}{dv}} = \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} = \sqrt{1 + 4x^2 + 4y^2} \]

\[
\int_{S_1} f(x, y, z) \, ds = \int_0^{2\pi} \int_0^2 f(r \cos(\theta), r \sin(\theta), 0) \, r \, dr \, d\theta \]

**Also you can use**
- \( x = r \cos(\theta) \)
- \( y = r \sin(\theta) \)
- \( z = 0 \)

\( \mathbf{r}_u \times \mathbf{r}_v \) = \( \left| \begin{array}{ccc} i & j & k \\ -v \sin(\theta) & r \cos(\theta) & 0 \\ v \cos(\theta) & r \sin(\theta) & 0 \end{array} \right| = \begin{pmatrix} r \sin(\theta) & -r \cos(\theta) & 0 \end{pmatrix} \]

\( \mathbf{r}_u \times \mathbf{r}_v \) = \( \sqrt{r^2 + (-v \sin(\theta))^2 + (r \cos(\theta))^2} = r \sqrt{1 + 4x^2 + 4y^2} \)

\( S_2 \) = the top \( z = 4 - x^2 - y^2 \)

**Parameterize:**
- \( x = x, y = y, z = 4 - x^2 - y^2 \)

\[
\left| \mathbf{r}_x \times \mathbf{r}_y \right| = \sqrt{1 + \left[ \frac{d}{dx} \frac{d}{dy} - \frac{d}{dx} \frac{d}{dy} \right] = \sqrt{1 + 4x^2 + 4y^2}} \]

\[
\int_{S_2} f(x, y, z) \, ds = \int_0^{\pi} \int_0^2 f(x, y, 4 - x^2 - y^2) \, r \, dr \, d\theta \]

**Use Polar**

\[
\int_{S} f(x, y, z) \, ds = \int_{S_1} f(x, y, z) \, ds + \int_{S_2} f(x, y, z) \, ds \]
Let $S$ be the upper hemisphere, centred at the origin, of radius 3, and inside the cylinder $x^2 + y^2 = 4$.

**ANS:**
The sphere $x^2 + y^2 + z^2 = 9$ and $-x^2 + y^2 = 4$ intersect when

$$\frac{z^2}{9} - \frac{x^2}{4} = 1$$

**PARAMETERIZE**

**OPTION 1:**
$x = x_if = y_i$

$z = \sqrt{9 - x^2 - y^2}$

**ABOVE**

$x^2 + y^2 \leq 1$

$$\int x \, dA = \sqrt{1 + \left(1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right)}$$

$$= \sqrt{1 + \left(\frac{-2x}{2\sqrt{9-x^2}}\right)^2 + \left(\frac{-2y}{2\sqrt{9-x^2}}\right)^2}$$

$$= \sqrt{1 + \frac{x^2 + y^2}{9 - x^2 - y^2}}$$

$$\iint f(x, y, z) \, dS$$

$$= \iint f(x, y, \sqrt{9-x^2-y^2}) \sqrt{1 + \frac{x^2 + y^2}{9 - x^2 - y^2}} \, dA$$

$$= \int_0^{2\pi} \int_0^\frac{\pi}{2} f(3\cos^2\phi, 3\sin^2\phi, 3\cos^2\phi) \cdot 3 \, d\phi \, d\theta$$
Let $S$ be the triangle with corners $P(1,0,0)$, $Q(0,5,0)$, $R(0,0,10)$.

**ANS**

**EQUATION FOR THE PLANE**

$	ext{PQ} = <-1,5,0>$  \  \  $	ext{PR} = <-1,0,10>$

$\text{normal} = \mathbf{n} = \text{PQ} \times \text{PR} = \begin{vmatrix} 
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-1 & 5 & 0 \\
-1 & 0 & 10 
\end{vmatrix} = \langle 50, 10, 5 \rangle$

$50(x-1) + 10y + 5z = 0$  \  \  divide by 5 to simplify

$10(x-1) + 2y + z = 0$

$z = -10x - 2y + 10$

**PROJECTION ONTO xy-plane** ($z=0$) $\Rightarrow$

$2y = -10x + 10$

$y = -5x + 5$

$x = 10, 5, y = 0, 5$

**PARAMETRIZE**

$x = x, \  \ y = y, \  \ z = -10x - 2y + 10$

$|F_x \times F_y| = \sqrt{1 + (-10)^2 + (-2)^2} = \sqrt{1 + 100 + 4} = \sqrt{105}$

$\int \int_S f(x, y, z) \, ds = \int \int_D f(x, y, -10x - 2y + 10) \sqrt{105} \, dA$

$\int_0^1 \int_0^{-5x+5} f(x, y, -10x - 2y + 10) \sqrt{105} \, dy \, dx$
Let $S$ be the part of the cylinder $x^2 + y^2 = 4$ that is above $z = 0$ and below $z = x^2 + 2$.

**ANS**

**Parameterize the cylinder**

\[
\begin{align*}
    x &= 2 \cos(u) \quad 0 \leq u \leq 2\pi \\
    y &= 2 \sin(u) \quad 0 \leq v \leq x^2 + 2 = 4 \cos^2(u) + 2 \\
    z &= v \\

    \mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix}
        \mathbf{i} & \mathbf{j} & \mathbf{k} \\
        2 \cos(u) & 2 \sin(u) & 0 \\
        0 & 0 & 1 
    \end{vmatrix} = 2 \cos(u) \mathbf{i} + 2 \sin(u) \mathbf{j} + 0 \mathbf{k} \\

    |\mathbf{r}_u \times \mathbf{r}_v| &= \sqrt{4 \cos^2(u) + 4 \sin^2(u) + 0} = 2 \\

    \iint_S f(x, y, z) \, dS &= \int_0^{2\pi} \int_0^{4 \cos^2(u) + 2} \int_0^2 f(2 \cos(u), 2 \sin(u), v) \, 2 \, dv \, du 
\end{align*}
\]
SURFACE INTEGRALS OF VECTOR FIELDS

\[ \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA \]

\[ \mathbf{F}(x, y, z) = \langle p(x, y, z), q(x, y, z), r(x, y, z) \rangle \]

\[ \text{Same surface set-up} \]

**Example**

Let \( S \) be the boundary of the solid between \( z = 4 - x^2 - y^2 \) and the xy-plane.

**Answer**

Compute the total outward flux.

- \( S_1 \) = the bottom
  
  \[ x = v \cos(u) \quad 0 \leq u \leq 2\pi \]
  
  \[ y = v \sin(u) \quad 0 \leq v \leq 2 \]
  
  \[ z = 0 \]

  \[ \mathbf{F}(v \cos(u), v \sin(u), 0) \cdot \langle 0, 0, -1 \rangle \\ \text{downward orientation which is outward} \]

  \[ \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^2 \mathbf{F}(v \cos(u), v \sin(u), 0) \cdot \langle 0, 0, -1 \rangle \, dv \, du \]

- \( S_2 \) = the top

  \[ x = x, y = y, z = 4 - x^2 - y^2 \]
  
  \[ \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -2x \\ 1 & 1 & -2y \end{vmatrix} = \langle 2x, 2y, 1 \rangle \]

  \[ \mathbf{F}(x, y, 4 - x^2 - y^2) \cdot \langle 2x, 2y, 1 \rangle \\ \text{upward orientation which is outward} \]

  \[ \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(x, y, 4 - x^2 - y^2) \cdot \langle 2x, 2y, 1 \rangle \, D \, dA \]

USE POLAR NEXT
Ex) Let $S$ be the upper hemisphere, centered at the origin, of radius 3 and inside the cylinder $x^2 + y^2 = 4$. (Upward orientation)

**ANS**

With spherical coordinates parameterization

$$x = 3 \sin \phi \cos \theta, \quad y = 3 \sin \phi \sin \theta, \quad z = 3 \cos \phi$$

$$0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \cos^{-1}(\frac{\sqrt{3}}{3})$$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ 3 \cos \phi \cos \theta & 3 \cos \phi \sin \theta & -3 \sin \phi \\ -3 \sin \phi \cos \theta & 3 \sin \phi \sin \theta & 0 \end{vmatrix}$$

**OUTWARD/UPWARD ORIENTATION**

$$\vec{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{\cos^{-1}(\frac{\sqrt{3}}{3})} \vec{F} \cdot (3 \sin \phi \cos \theta, 3 \sin \phi \sin \theta, 3 \cos \phi) \cdot (9 \sin^2 \phi \cos \theta, 9 \sin^2 \phi \sin \theta, 9 \sin \phi \cos \phi) \, d\phi \, d\theta$$

Ex) Let $S$ be the triangle with corners $P(1, 9, 0), Q(9, 5, 0), R(9, 9, 10)$. (Upward orientation)

**ANS**

Equation for the plane

$$z = -10x - 2y + 10$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -10 \\ 0 & 1 & -2 \end{vmatrix} = (10, 2, 10)$$

**UPWARD ORIENTATION**

$$\vec{F} \cdot d\mathbf{S} = \int_{-5}^{5} \int_{-\sqrt{5-x^2}}^{\sqrt{5-x^2}} \vec{F} \cdot (x, y, -10x - 2y + 10) \cdot (10, 2, 10) \, dy \, dx$$
Ex) Let $S$ be the part of the cylinder $x^2 + y^2 = 4$ that is above $z = 0$ and below $z = x^2 + 2$. (OUTWARD ORIENTATION)

Ans

$x = 2 \cos (u)$ \hspace{1cm} 0 \leq u \leq \pi$

$y = 2 \sin (u)$ \hspace{1cm} 0 \leq v \leq x^2 + 2 = 4 \cos^2(u) + 1$

$z = v$ \hspace{1cm}$\text{position vector for circle points outward}$

$F \cdot \mathbf{n} = \langle 2 \cos(u), 2 \sin(u), 0 \rangle$

$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_D \mathbf{F} \cdot \mathbf{n} \, dA$

$= \int_{0}^{2\pi} \int_{0}^{4 \cos^2(u) + 1} \mathbf{F} \cdot \langle 2 \cos(u), 2 \sin(u), 0 \rangle \, dv \, du$