16.6 Parameterizing Surfaces

Recall that \( \mathbf{r}(t) = (x(t), y(t), z(t)) \) with \( a \leq t \leq b \) gives a parameterization for a curve \( C \). In section 16.2-16.4, we learned how to make measurements along curves for scalar and vector fields by using line integrals \( \int_C \). We computed these line integrals by first finding parameterizations (unless special theorems apply).

In a similar way, we will parameterize a surface \( S \) using

\[ \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)), \]

where \((u, v)\) are constrained to some region \( D \) in the \( uv\)-plane. In section 16.7-16.9, we learned how to make measurements across surfaces for scalar and vector fields by using surface integrals \( \int_S \). We will compute these surface integrals by first finding parameterizations (and later we will learn theorems that apply in special cases).

For now, let’s focus on parameterization.

**Questions:** Find a parameterization for each surface:

1. The part of the surface \( z = 10 \) that is above the square \(-1 \leq x \leq 1, -2 \leq y \leq 2\).
2. The part of the surface \( x - y + z = 4 \) that is within the cylinder \( x^2 + y^2 = 9 \).
3. The part of the surface \( z = x^2 + y^2 \) that is above the region in the \( xy\)-plane given by \( 0 \leq x \leq 1, 0 \leq y \leq x^2 \).
4. The part of the paraboloid \( y = 9 - x^2 - z^2 \) that is on the positive \( y \) side of the \( xz\)-plane.
5. The part of the circular cylinder \( x^2 + y^2 = 4 \) that is between the planes \( z = 1 \) and \( z = 5 \).
6. The upper hemisphere of the sphere \( x^2 + y^2 + z^2 = 9 \).
7. The entire sphere \( x^2 + y^2 + z^2 = 16 \).
8. The surface of revolution given by rotating the region bounded by \( y = x^3 \) for \( 0 \leq x \leq 2 \) about the \( x\)-axis.
9. Find the parameterization for all three sides of the solid object within \( x^2 + y^2 = 1 \), above \( z = 0 \) and below \( z = 5 - x \) shown here (ignore the curve):
Solutions:

1. Notes: The parameterization is already given!
   \( \mathbf{r}(u, v) = (u, v, 10) \), (I am just letting \( x = u \) and \( y = v \)).
   You could also just leave them as \( x \) and \( y \) and give the parameterization as:
   \( \mathbf{r}(x, y) = (x, y, 10) \) with \( -1 \leq x \leq 1, -2 \leq y \leq 2 \).

2. Notes: The surface can easily be solve for \( z \) in terms of \( x \) and \( y \).
   \( \mathbf{r}(u, v) = (u, v, 4 - u + v) \), (Letting \( x = u \) and \( y = v \), again). Also can be written as:
   \( \mathbf{r}(x, y) = (x, y, 4 - x + y) \) for points \((x, y)\) inside the circular region \( x^2 + y^2 \leq 4 \) (which we will do with polar when we get to the integral).

3. \( \mathbf{r}(x, y) = (x, y, x^2 + y^2) \) for points \((x, y)\) inside the region given by \( 0 \leq x \leq 1, 0 \leq y \leq x^2 \) (again, we will account for this in the integral later).

4. Notes: This time it is easiest to give \( y \) in terms of \( x \) and \( z \).
   \( \mathbf{r}(x, z) = (x, 9 - x^2 - z^2, z) \) for points \((x, z)\) within the region when \( y \geq 0 \) on the surface. That
   would be when \( 9 - x^2 - z^2 \geq 0 \) which would be the circular region \( x^2 + z^2 \leq 9 \).

5. Notes: This is different from the previous cases, because one variable is ‘missing’ from the surface we wish to describe. That means \( z \) can be anything and we should make it one of our parameters. Then we need to find a parameterization for the other two variables. Look to use Sine and Cosine!
   \( \mathbf{r}(u, v) = (2 \cos(u), 2 \sin(u), v) \), (This time, I am letting \( x = 2 \cos(u), y = 2 \sin(u) \) and \( z = v \)).
   We need \( 1 \leq v \leq 5 \) from the given condition.
   And we need \( 0 \leq u \leq 2\pi \) to go all the way around the cylinder.

6. Notes: This could be done in a couple ways. Here are two different parameterizations:
   
   (a) We could just get \( z \) in terms of \( x \) and \( y \). That would give \( z = \sqrt{9 - x^2 - y^2} \) for the upper hemisphere. Giving the parameterization
   \( \mathbf{r}(x, y) = (x, y, \sqrt{9 - x^2 - y^2}) \), where \((x, y)\) come from the region that corresponds to \( z \geq 0 \)
   in the surface equation, so \( 9 - x^2 - y^2 \geq 0 \), which is the circular region \( x^2 + y^2 \leq 9 \).
   
   (b) We could use spherical coordinators. Notice that the radius of the sphere, \( \rho = 3 \), is fixed.
   \( \mathbf{r}(\phi, \theta) = (3 \sin \phi \cos \theta, 3 \sin \phi \sin \theta, 3 \cos \phi) \), where \((\phi, \theta)\) satisfy \( 0 \leq \phi \leq \pi/2 \) and \( 0 \leq \theta \leq 2\pi \).

7. Notes: I would use spherical coordinates here (or break the problem into two parts; upper and lower hemisphere). Again the radius of the sphere, \( \rho = 4 \), is fixed.
   \( \mathbf{r}(\phi, \theta) = (4 \sin \phi \cos \theta, 4 \sin \phi \sin \theta, 4 \cos \phi) \), where \((\phi, \theta)\) would satisfy \( 0 \leq \phi \leq \pi \) and \( 0 \leq \theta \leq 2\pi \).

8. Notes: For a surface of revolution about the \( x \)-axis, there is a circle of radius \( f(x) \) about each
   value of \( x \). So we can parameterize each of those circles to get
   \( \mathbf{r}(u, v) = (u, f(u) \cos(v), f(u) \sin(v)) \), so I am just replacing \( x = u \) and then parameterizing the circle. The range of values would be \( 0 \leq u \leq 2, \) and \( 0 \leq v \leq 2\pi \).

9. Here is a parameterization for each side:
   
   (a) Bottom: \( \mathbf{r}(x, y) = (x, y, 0) \), where \((x, y)\) are in the region \( x^2 + y^2 \leq 1 \).
   
   (b) Top: \( \mathbf{r}(x, y) = (x, y, 5 - x) \), where \((x, y)\) are in the region \( x^2 + y^2 \leq 1 \).
   
   (c) Sides: \( \mathbf{r}(u, v) = (\cos(u), \sin(u), v) \), where \((u, v)\) satisfy \( 0 \leq u \leq 2\pi \) and \( 0 \leq v \leq 5 - \cos(u) \).
   (I got the last bound because \( z \) is always between 0 and \( 5 - x \) and in this parameterization \( z = v \) and \( x = \cos(u) \)).
Surface Area

After parameterizing, our next step will be to give an expression for surface area. Way back in 15.6, we already learned that the surface area for a surface parameterized by \( \mathbf{r}(x,y) = \langle x, y, f(x,y) \rangle \) over a region \( D \) is given by \( \iint_D 1 \, dS \), where

\[
dS = |\mathbf{r}_x \times \mathbf{r}_y| \, dA = \sqrt{(f_x)^2 + (f_y)^2 + 1} \, dA.
\]

That was only for those particular parameterizations. But the same general analysis applies. For a parameterization, \( \mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle \). We have

\[
\mathbf{r}_u = \langle x_u, y_u, z_u \rangle = \text{a tangent vector to the surface in the } u\text{-direction.}
\]

\[
\mathbf{r}_v = \langle x_v, y_v, z_v \rangle = \text{a tangent vector to the surface in the } u\text{-direction.}
\]

We then get several facts:

1. \( \mathbf{r}_u \) and \( \mathbf{r}_v \) together determine the tangent plane at a given point (because they are both ‘on’ this plane). So \( \mathbf{r}_u \times \mathbf{r}_v \) would be a normal vector for the surface at a given point (and a normal for the tangent plane at that point).

2. If a small change in \( u \) and a small change in \( v \) are made, \( \Delta u \) and \( \Delta v \), respectively, then we can estimate the resulting change in surface area by

\[
\Delta S = |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v.
\]

As \( \Delta u \) and \( \Delta v \) go to zero, this gets more precise and we write the surface area differential for this relationship as

\[
dS = |\mathbf{r}_u \times \mathbf{r}_v| \, dudv.
\]

3. From 15.6, the surface area of the surface is given by

\[
\text{Surface area} = \iint_D dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA
\]

4. Some shortcuts:

(a) For a parameterization of the form \( \mathbf{r}(x,y) = \langle x, y, f(x,y) \rangle \), we get

\[
\mathbf{r}_x \times \mathbf{r}_y = \langle -f_x, -f_y, 1 \rangle
\]

\[
|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{(f_x)^2 + (f_y)^2 + 1}
\]

(b) For a parameterization of the form \( \mathbf{r}(\phi, \theta) = \langle a \sin \phi \cos \theta, a \sin \phi \sin \theta, a \cos \phi \rangle \), we get

\[
\mathbf{r}_x \times \mathbf{r}_y = \langle a^2 \sin^2 \phi \cos \theta, a^2 \sin^2 \phi \sin \theta, a^2 \cos^2 \phi \rangle
\]

\[
|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2 \sin \phi
\]