Math 310 Definitions and Theorems from Chapters 1-3

**Essential Definitions**

1. \( n \in \mathbb{Z} \) is even \( \Rightarrow \ n = 2k \) for some \( k \in \mathbb{Z} \).
2. \( n \in \mathbb{Z} \) is odd \( \Rightarrow \ n = 2k + 1 \) for some \( k \in \mathbb{Z} \).
3. \( x \in A \cup B \Rightarrow x \in A \) or \( x \in B \).
4. \( x \in A \cap B \Rightarrow x \in A \) and \( x \in B \).
5. \( x \in A - B \Rightarrow x \in A \) and \( x \notin B \).
6. \( x \in A^c \Rightarrow x \notin A \).
7. \( A \subseteq B \Rightarrow (x \in A \text{ implies } x \in B) \).
8. \( y \in f(S) \Rightarrow \text{ there exists an } x \in S \text{ such that } y = f(x) \).
9. \( x \in I_f(S) \Rightarrow \text{ there exists } y \in S \text{ such that } f(x) = y \).
10. \( f \) is bounded \( \Rightarrow \text{ there exists an } M \in \mathbb{R} \text{ such that } |f(x)| \leq M \text{ for all } x \in \mathbb{R} \).

**Essential Theorems**

1. (de Morgan’s Laws)
   - \((A \cup B)^c = A^c \cap B^c\)
     Another way to state this is: \( x \notin A \cup B \) if and only if \( x \notin A \) and \( x \notin B \).
   - \((A \cap B)^c = A^c \cup B^c\)
     Another way to state this is: \( x \notin A \cap B \) if and only if \( x \notin A \) or \( x \notin B \).

2. (Logically equivalent statements) \((P \Rightarrow Q) \iff (\neg Q \Rightarrow \neg P) \iff \neg(P \land \neg Q)\) These logical equivalences characterize three main proof techniques (in order from left to right: Direct Proof, Contrapositive, and Negation).

3. (Inequalities)
   - (Triangle Inequality) \( \forall x, y \in \mathbb{R}, |x + y| \leq |x| + |y| \).
   - (AGM Inequality) \( \forall x, y \in \mathbb{R}, (a) 2xy \leq x^2 + y^2 \text{ and } (b) xy \leq \left( \frac{x^2 + y^2}{2} \right)^2 \).
     In addition, if \( x, y \geq 0 \), then \( \sqrt{xy} \leq \frac{x+y}{2} \).
   - (Frequently used axioms)
     - If \( x \leq y \) and \( u \leq v \), then \( x + u \leq y + v \). (i.e. we can add inequalities).
     - If \( 0 \leq x \leq y \) and \( 0 \leq u \leq v \), then \( xu \leq yv \). (i.e. we can multiply inequalities provided all values are positive. Note: we can handle negative values as well, but be half to be careful about when to switch the direction of the inequality.)
Main Proof Techniques / Proof Templates

Here is what each proof technique should look like. That is, these are the templates that you are filling in when you give a proof.

1. **Any Direct Proof** \((P \Rightarrow Q)\)
   
   *Theorem* \(P\) implies \(Q\).
   
   *Proof* Let \(P\) be true.
   
   ;
   
   (Here you write out the definitions that appear in \(P\) and you try to show using logical deductions that the definitions in \(Q\) are satisfied)
   
   ;
   
   Thus, \(Q\) is true. ■

2. **Contrapositive** \((\neg Q \Rightarrow \neg P)\)
   
   *Theorem* \(P\) implies \(Q\)
   
   *Proof* We prove the contrapositive. Let \(\neg Q\) be true.
   
   ;
   
   (Here you write out the definitions that appear in \(\neg Q\) and you try to show using logical deductions that the definitions in \(\neg P\) are satisfied)
   
   ;
   
   Thus, \(\neg P\) is true. ■

3. **Contradiction** \((\neg(P \land \neg Q))\)
   
   *Theorem* \(P\) implies \(Q\)
   
   *Proof* We assume the negation in order to get a contradiction. Let \(P\) and \(\neg Q\) both be true.
   
   ;
   
   (Here you write out the definitions that appear in \(P\) and in \(\neg Q\) and you try to show using logical deductions that a contradiction arises)
   
   ;
   
   Thus, we have arrived at a contradiction \((\rightarrow \leftarrow)\). ■
Special Proof Techniques / Proof Templates

1. Sets

- **Theorem** $A \subseteq B$
  
  **proof** Let $x \in A$.

  (Here you write out *exactly* what it means for $x \in A$ using all the definitions that are contained in the left hand side. Then show that $x$ satisfies the definitions that appear on the right hand side)

  Thus, $x \in B$. ■

- **Theorem** $A = B$
  
  **proof** Let $x \in A$.

  Thus, $x \in B$. So $A \subseteq B$.
  
  Let $x \in B$.

  Thus, $x \in A$. So $B \subseteq A$. ■

2. **Induction** The following methods can be used to prove statements for all $n \in \mathbb{N}$.

  Important notes: In the inductive step it is often useful to use one of the following two techniques:

  (a) For a sum, it is always true that $\sum_{i=1}^{k+1} a_i = a_{k+1} + \sum_{i=1}^{k} a_i$. This is useful in induction because we can use the inductive hypothesis on the sum up to $k$.

  (b) For a product, it is always true that $\prod_{i=1}^{k+1} a_i = a_{k+1} \prod_{i=1}^{k} a_i$. We can use the inductive hypothesis on the product up to $k$.

  (c) For a power, it is always true that $y^{k+1} = yy^k$. We can use the induction hypothesis on the $y^k$ part.
• Basic Induction

**Theorem** For all \( n \in \mathbb{N} \), \( P(n) \) is true.

**proof** We use induction on \( n \).

Base Step: For \( n = 1 \), we show that \( P(1) \) is true.

(This usually involves plugging in one to two sides of a formula to show that they are equal).

Induction Step: Assume \( P(k) \) is true for some \( k \geq 1 \).

(Show that \( P(k+1) \) is true by somehow breaking up the problem so that you can use \( P(k) \). At some point in here you must use the inductive hypothesis and you need to tell me when you did so.)

Thus, \( P(k+1) \) is true. ■

• Strong Induction

**Theorem** For all \( n \in \mathbb{N} \), \( P(n) \) is true.

**proof** We use strong induction on \( n \).

Base Step: For \( n = 1 \), we show that \( P(1) \) is true.

(This usually involves plugging in one to two sides of a formula to show that they are equal, if you are using a formula that involves the previous 2 terms then you need to also show that \( P(2) \) is true. Similarly for 3 terms, etc.).

Induction Step: Assume \( P(1), P(2), ..., \) and \( P(k) \) are all true for some \( k \geq 1 \).

(Show that \( P(k+1) \) is true by somehow breaking up the problem so that you can use the previous values \( P(1), P(2), ..., \) and \( P(k) \). At some point in here you must use the inductive hypothesis and you need to tell me when you did so.)

Thus, \( P(k+1) \) is true. ■