Math 310

Final Solutions

1. (a) Negation
   If it is snowing AND (Billy is barefoot OR Phil is eating ice cream).

   (b) (i) Contrapositive
   If \( \gcd(4,a) = 1 \) and \( 4 \not\mid n \), then \( 4 \not\mid an \).

   (ii) True

2. Thm \( \frac{1}{i(i+1)} = \frac{n}{n+1}, \forall n \in \mathbb{N} \)
   \[ \text{pf: Induction on } n \]
   Base Step: For \( n = 1 \), \( \frac{1}{1(1+1)} = \frac{1}{2} \).
   Ind Step: Assume \( \frac{k}{(k+1)(k+2)} = \frac{1}{k+1} \) for some \( k \in \mathbb{N} \).
   Then \( \frac{k+1}{(k+1)(k+2)} = \frac{1}{k+1} + \frac{k}{(k+1)(k+2)} \) (by in. hyp.)
   \( = \frac{1+k(k+2)}{[(k+1)(k+2)]} \)
   \( = \frac{k^2+2k}{[(k+1)(k+2)]} \)
   \( = \frac{(k+1)^2}{[(k+1)(k+2)]} \)
   \( = \frac{k+1}{k+2} = \frac{k}{k+1} + 1 \)

3. Reflexive
   NO
   Symmetric
   YES
   Transitive
   NO

\[ (x-x)^2 = 0 \neq 1, \]
\[ (x-y)^2 = 1 \iff (y-x)^2 = 1, \]
\[ (x-y)^2 = 1, (y-z)^2 = 1 \]
\[ \iff x-y = \pm 1, y-z = \pm 1 \]
\[ \iff x = y \pm 1, y = z \pm 1 \]
But \( x \) could be \( \pm z+2 \)
It does not have to be the case that \( z+1 \) or \( z+0 \) or \( z-2 \)
\[ x=\frac{z}{2} = \pm 1, \]
\[ (2-0)^2 = 1, (1-0)^2 = 1 \]
But \( (2-0)^2 = 4 \)

Counterexample
\[ (2-0)^2 = 1, (1-0)^2 = 1 \]
4. Thm
If \( a+b-c \equiv 0 \pmod{n} \)
then \( b \equiv c \pmod{\gcd(a,n)} \).

pf
Let \( d = \gcd(a,n) \). Then \( d|a \) and \( d|n \).
Since \( a+b-c \equiv 0 \pmod{n} \), by def'n,
\[ n \mid a+b-c. \]
Thus, \( a+b-c = nk \) for some \( k \in \mathbb{Z} \).
Also, \( a=du \) and \( n=dv \) for some \( u,v \in \mathbb{Z} \).
Hence, \( b-c = nk - a = d(kv-u) \), which implies
\[ d \mid b-c. \]
Ergo, \( b \equiv c \pmod{d} \).

5. Thm
\( S = \{4n \} \) \( |S| = 3n+1 \)

pf We use the PHP.
objects: Elements of \( S \) \( (3n+1 \) of them)\nclasses: Nonoverlapping quadruplets,
\[ \{1,2,3,4\}, \{5,6,7,8\}, \ldots, \{4n-3,4n-2,4n-1,4n\}. \]
Since there are \( n \) classes and
\( 3n+1 > kn \) with \( k=3 \), the general
pigeonhole principle implies that some
class must contain four elements of \( S \).
Thus, \( S \) contains a number that \( \not\equiv 0 \pmod{4} \).
(Since each class has a number divisible by 4.)
6. (a) $9 \mid R_n$ iff $9$ divides the sum of the digits of $R_n$.

This only happens when $3 \mid n$.

(b) **Thm** $R_n \equiv 0 \pmod{11}$ iff $n$ is even.

**PF** 
Observe $10 \equiv -1 \pmod{11}$

So $10^n \equiv (-1)^n \equiv \begin{cases} 1 & \text{if } n \text{ is even;} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$

Note: $R_n = 3 + 3 \cdot 10 + 3 \cdot 10^2 + \cdots + 3 \cdot 10^{n-1}$

$= 3 - 3 + 3 - 3 + \cdots + (-1)^{n-1} \cdot 3$

$\equiv \begin{cases} 0 & \text{if } n \text{ is even;} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$

Thus $R_n \equiv 0 \pmod{11}$ iff $n$ is even.

7. (a) **Thm** If $g$ is an injection, then 

$g(R) \cap g(S) \subseteq g(R \cap S \cap T)$.

**PF** 
Let $y \in g(R) \cap g(S) \cap g(T)$.

Then $y = g(x_1)$ for some $x_1 \in R$ AND 
$y = g(x_2)$ for some $x_2 \in S$ AND 
$y = g(x_3)$ for some $x_3 \in T$.

Since $g$ is injective, $x_1 = x_2 = x_3 = x$.

So $x \in R \cap S \cap T$ and $y = f(x)$.

Hence, $y \in g(R \cap S \cap T)$.

(b) **Thm** If $g$ is injective and $f$ is strictly increasing then $\hat{h} = f \circ g$ is injective.

**PF** 
Assume $h(x_1) = h(x_2)$ for some $x_1, x_2 \in A$.

Then $f(g(x_1)) = f(g(x_2)) \Rightarrow g(x_1) = g(x_2)$

because $f$ is injective.

If $x_1 < x_2$, then $g(x_1) < g(x_2)$ \Rightarrow \epsilon.

If $x_1 > x_2$, then $g(x_1) > g(x_2) \Rightarrow \epsilon$. Thus, $x_1 = x_2$. \(/\)}
8 (a) Thm: \( p | a \iff p^3 | a^3 \)

pf: 
- If \( p | a \), then \( a = pk \) for some \( k \in \mathbb{Z} \).
  
  Then \( a^3 = (pk)^3 = p^3k^3 \).
  
  \( \Rightarrow \) \( p^3 | a^3 \).

- If \( p^3 | a^3 \), then \( a^3 = kp^3 = p^2kp \) for some \( k \in \mathbb{Z} \).
  
  Thus, \( p^2 | a \).
  
  \( \Leftarrow \) If \( p^2 | a \), then \( a^3 = kp^2p^2k \) for some \( k \in \mathbb{Z} \).
  
  Thus, \( p^2 | a^3 \).
  
  In the prime factorization of \( a \), \( p \) must be contained.
  
  Hence, \( p | a \).

(b) Thm: \( \sqrt[3]{2} \) is an irrational number.

pf: Assume \( \sqrt[3]{2} \) is a rational number.

Then \( \frac{a}{b} = \sqrt[3]{2} \) with \( \gcd(a,b) = 1 \) and \( b \neq 0 \).

Hence \( 2 = a^3/b^3 \Rightarrow 2b^3 = a^3 \).

So, 2 must divide \( a^3 \) (a must contain a factor of 2).

Thus, \( 2 | a^3 \Rightarrow 2 | a \Rightarrow 2b^3 = a^3 \Rightarrow b^3 = 1 \Rightarrow b \) is even.

So \( \gcd(a,b) > 1 \).

\( \Rightarrow \) \( \sqrt[3]{2} \) is an irrational number.