1. (7 total points)

(a) (4 points) A tank is filled with 100 gal of solution which initially contains 50 g of sugar. Solution enters the tank at a rate of 200 gal/h. Said solution contains e^{-t} g/gal of sugar after t hours. The well-stirred solution leaves the tank at 200 gal/h. Use the **Laplace transform** to determine the amount of sugar in the tank after any time $t > 0$.

The amount of sugar Q satisfies

$$Q' = \text{rate in} - \text{rate out} = 200e^{-t} - 200\frac{Q}{100} = 200e^{-t} - 2Q, \qquad Q(0) = 50.$$

Apply the Laplace transform to deduce

$$\frac{200}{s+1} = \mathscr{L}[200e^{-t}] = \mathscr{L}[Q' + 2Q] = s\mathscr{L}[Q] - Q(0) + 2\mathscr{L}[Q] = (s+2)\mathscr{L}[Q] - 50,$$

so

$$\mathscr{L}[Q] = \frac{200}{(s+1)(s+2)} + \frac{50}{s+2}.$$

Now compute the partial fraction decomposition of the first term:

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} \Rightarrow 1 = (A+B)s + (A+2B),$$

so $0 = A+B$ and $1 = A+2B$ which implies $A = 1$ and $B = -1$. Thus

$$\begin{aligned} Q &= \mathscr{L}^{-1}\left[\frac{200}{(s+1)(s+2)} + \frac{50}{s+2}\right] \\ &= 200\left(\mathscr{L}^{-1}\left[\frac{1}{s+1}\right] - \mathscr{L}^{-1}\left[\frac{1}{s+2}\right]\right) + 50\mathscr{L}^{-1}\left[\frac{1}{s+2}\right] \\ &= 200e^{-t} - 150e^{-2t}. \end{aligned}$$

(b) (3 points) Suppose the solution flowing out of the tank mentioned in part (a) flows into a second tank. Initially, the second tank contains 50 gal of solution with an unknown initial amount of sugar, and the solution leaves the tank at a rate of 200 gal/h. Use a method of your choice to determine the amount of sugar in this second tank at any time $t > 0$. *Note: Your answer will contain an unspecified constant.*

The amount of sugar in tank 2, denotes by R satisfies

$$R' = \text{rate in} - \text{rate out} = 2Q - 200\frac{R}{50} = 400e^{-t} - 300e^{-2t} - 4R;$$

we have used part (a) to replace Q. Let us solve this differential equation using integrating factors (you can also use the Laplace transform):

$$R'\mu + 4R\mu = \left(400e^{-t} - 300e^{-2t}\right)\mu.$$

so the integrating factor μ satisfies $\mu' = 4\mu$. Thus $\mu = e^{4t}$ is one integrating factor, and

$$(Re^{4t})' = R'e^{4t} + 4Re^{4t} = \left(400e^{-t} - 300e^{-2t}\right)e^{4t} = 400e^{3t} - 300e^{2t}.$$

Thus

$$Re^{4t} = \int 400e^{3t} - 300e^{2t}\,\mathrm{d}t = \frac{400}{3}e^{3t} - \frac{300}{2}e^{2t} + C,$$

so we conclude

$$R = \frac{400}{3}e^{-t} - 150e^{-2t} + Ce^{-4t}.$$

2. (6 points) A mass weighing 1 lb stretches a spring 2 ft. The mass is in a perfect vacuum so there is no air resistance. Suppose that an external force of $\frac{1}{16}\cos(4t)$ lb is applied to the mass after t seconds. Finally, suppose that the mass is displaced by 2 ft upward and then set in a motion with a speed of 1 ft/s in the downward direction. Use the **tools from chapter 3** in the textbook (solution formula, method of undetermined coefficients) to determine the position of the mass at any time $t > 0$.

Since $1 = w = mg = 32m$, it follows that $m = \frac{1}{32}$. Next, we know that $mg - kL = 0$, so $k = \frac{mg}{L} = \frac{1}{2}$. Since there is no damping, $\gamma = 0$. Thus

$$\frac{1}{32}u'' + \frac{1}{2}u = \frac{1}{16}\cos(4t) \Rightarrow u'' + 16u = 2\cos(4t)$$

The initial conditions are $u(0) = -2$ and $u'(0) = 1$.

We first determine the solution to the homogeneous equation u_h. The characteristic polynomial is $r^2 + 16 = 0$ which has zeros $r = \pm 4i$. By the solution formula,

$$u_h = C\cos(4t) + D\sin(4t).$$

Next, we determine a particular solution to the non-homogeneous equation. Our initial guess is of the form $A\cos(4t) + B\sin(4t)$, but that is a solution to the homogeneous equation. We update the guess to

$$u_c = At\cos(4t) + Bt\sin(4t).$$

Now,

$$\begin{aligned} u_c' &= A\cos(4t) - 4At\sin(4t) + B\sin(4t) + 4Bt\cos(4t) \\ u_c'' &= -4A\sin(4t) - 4A\sin(4t) - 16At\cos(4t) + 4B\cos(4t) + 4B\cos(4t) - 16B\sin(4t) \end{aligned}$$

Thus

$$\begin{aligned} 2\cos(4t) &= u_c'' + 16u_c \\ &= -8A\sin(4t) - 16At\cos(4t) + 8B\cos(4t) - 16B\sin(4t) \\ &\quad +16\left(At\cos(4t) + Bt\sin(4t)\right) \\ &= -8A\sin(4t) + 8B\cos(4t), \end{aligned}$$

so $A = 0$ and $B = \frac{1}{4}$. The full solution is therefore

$$u = C\cos(4t) + D\sin(4t) + \frac{1}{4}t\sin(4t).$$

Finally, we use the initial conditions to determine C and D: Clearly $-2 = u(0) = C$, and

$$1 = u'(0) = 8\sin(4t) + 4D\cos(4t) + \frac{1}{4}\sin(4t) + t\cos(4t)\Big|_{t=0} = 4D,$$

so $D = \frac{1}{4}$. We conclude that

$$u = -2\cos(4t) + \frac{1}{4}\sin(4t) + \frac{1}{4}t\sin(4t).$$

3. (0 total points) An electrical circuit contains a resistor of $1\,\Omega$ and a capacitor. Its capacitance is varied over time and equals $(1+\cos(t))^{-1}$ F after t seconds. A generator impresses a voltage of $E(t) = e^{-\sin(t)}$ V after t seconds. Assume the initial charge in the circuit is $5\,$C.

   (a) (4 points) Determine the charge at any time $t > 0$.

The charge Q satisfies

$$Q' + \frac{1}{(1+\cos(t))^{-1}}Q = Q' + (1+\cos(t))Q = e^{-\sin(t)}, \qquad Q(0) = 5.$$

To solve it, multiply by an integrating factor μ,

$$Q'\mu + Q(1+\cos(t))\mu = e^{-\sin(t)}\mu$$

so μ has to satisfy $\mu' = \mu(1+\cos(t))$. Thus

$$\log(\mu) = \int \frac{\mathrm{d}\mu}{\mu} = \int 1+\cos(t)\mathrm{d}t = t + \sin(t),$$

and one solution is $\mu = e^{t+\sin(t)}$. Therefore,

$$\left(Qe^{t+\sin(t)}\right)' = Q'e^{t+\sin(t)} + Q(1+\cos(t))e^{t+\sin(t)} = e^{-\sin(t)}e^{t+\sin(t)} = e^t.$$

Integrating yields

$$Qe^{t+\sin(t)} = \int e^t \mathrm{d}t = e^t + C.$$

Evaluate this equation at zero to deduce

$$5 = Q(0)\cdot 1 = 1 + C \Rightarrow C = 4,$$

and

$$Q = e^{-\sin(t)} + 4e^{-t-\sin t}.$$

   (b) (1 point) Determine the limit as $t \to \infty$ of the ratio of the charge and the impressed voltage.

$$\lim_{t\to\infty} \frac{Q(t)}{E(t)} = \lim_{t\to\infty} \frac{e^{-\sin(t)} + 4e^{-t-\sin t}}{e^{-\sin(t)}} = \lim_{t\to\infty} 1 + 4e^{-t} = 1.$$

4. (7 points) An electrical circuit is described by the equation

$$Q'' + Q = E$$

where Q is the charge and E is the impressed voltage. Suppose that the circuit is initially closed, i e. $Q(0) = Q'(0) = 0$, and that a generator impressing a voltage of $E = 5e^{-t}\sin(t)$ V is connected to the circuit at time $t = \pi$. Use the **Laplace transform** to determine the charge at any time $t > 0$. *Hint: All the identities you need are on the note sheet.*

The impressed voltage is

$$E(t) = \begin{cases} 0 & 0 < t \leq \pi \\ 5e^{-t}\sin(t) & t > \pi \end{cases} = H_\pi 5e^{-t}\sin(t),$$

so the charge satisfies

$$Q'' + Q = 5H_\pi e^{-t}\sin(t), \qquad Q(0) = 0, \quad Q'(0) = 0.$$

Apply the Laplace transform to deduce

$$\mathscr{L}[Q'' + Q] = s^2\mathscr{L}[Q] - sQ(0) - Q(0) + \mathscr{L}[Q] = (s^2+1)\mathscr{L}[Q].$$

Since $e^{-t} = e^{-\pi}e^{-(t-\pi)}$ and $\sin(t) = -\sin(t-\pi)$, we get

$$\begin{aligned} \mathscr{L}\left[H_\pi 5e^{-t}\sin(t)\right] &= -5e^{-\pi}\mathscr{L}\left[H_\pi e^{-(t-\pi)}\sin(t-\pi)\right] \\ &= -5e^{-\pi}\mathscr{L}\left[H_\pi\left(e^{-t}\sin(t)\right)_\pi\right] \\ &= -5e^{-\pi}e^{-\pi s}\mathscr{L}[e^{-t}\sin(t)] \\ &= -5e^{-\pi}e^{-\pi s}\frac{1}{(s+1)^2+1}. \end{aligned}$$

Thus

$$\mathscr{L}[Q] = -5e^{-\pi}e^{-\pi s}\frac{1}{((s+1)^2+1)(s^2+1)}$$

and

$$\begin{aligned} Q &= \mathscr{L}^{-\pi}\left[-5e^{-1}e^{-\pi s}\frac{1}{((s+1)^2+1)(s^2+1)}\right] \\ &= -5e^{-\pi}\mathscr{L}^{-\pi}\left[e^{-\pi s}\frac{1}{(s^2+2s+2)(s^2+1)}\right] \\ &= -5e^{-\pi}H_\pi\left(\mathscr{L}^{-1}\left[\frac{1}{(s^2+2s+2)(s^2+1)}\right]\right)_\pi \end{aligned}$$

To determine this inverse Laplace transform, we compute the partial fraction decomposition:

$$\frac{1}{(s^2+2s+2)(s^2+1)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+1}$$

Thus

$$\begin{aligned} 1 &= (As+B)(s^2+1) + (Cs+D)(s^2+2s+2) \\ &= (A+C)s^3 + (B+2C+D)s^2 + (A+2C+2D)s + (B+2D) \end{aligned}$$

so $A+C=0$, $B+2C+D=0$, $A+2C+2D=0$, and $B+2D=1$. The first equation implies $A=-C$, so the third equation can be written as $C=-2D$. Plug this into the second equation to deduce $B = 3D$. This and the fourth equation imply $D = \frac{1}{5}$, so $C = -\frac{2}{5}$, $B = \frac{3}{5}$, and $A = \frac{2}{5}$. Thus

$$\begin{aligned}
Q &= -5e^{-\pi}H_\pi\left(\mathscr{L}^{-1}\left[\frac{1}{(s^2+2s+2)(s^2+1)}\right]\right)_\pi \\
&= -5e^{-\pi}H_\pi\left(\mathscr{L}^{-1}\left[\frac{\frac{2}{5}s+\frac{3}{5}}{s^2+2s+2}+\frac{-\frac{2}{5}s+\frac{1}{5}}{s^2+1}\right]\right)_\pi \\
&= -5e^{-\pi}H_\pi\left(\frac{2}{5}\mathscr{L}^{-1}\left[\frac{s+1}{(s+1)^2+1}\right]+\frac{1}{5}\mathscr{L}^{-1}\left[\frac{1}{(s+1)^2+1}\right]-\frac{2}{5}\mathscr{L}^{-1}\left[\frac{s}{s^2+1}\right]\right. \\
&\quad \left.+\frac{1}{5}\mathscr{L}^{-1}\left[\frac{1}{s^2+1}\right]\right)_\pi \\
&= -5e^{-\pi}H_\pi\left(\frac{2}{5}e^{-t}\cos(t)+\frac{1}{5}e^{-t}\sin(t)-\frac{2}{5}\cos(t)+\frac{1}{5}\sin(t)\right)_\pi \\
&= -5e^{-\pi}H_\pi\left(\frac{2}{5}e^{-(t-\pi)}\cos(t-\pi)+\frac{1}{5}e^{-(t-\pi)}\sin(t-\pi)-\frac{2}{5}\cos(t-\pi)+\frac{1}{5}\sin(t-\pi)\right)
\end{aligned}$$

This can be simplified to

$$Q = H_\pi\left(2e^{-t}\cos(t)+e^{-t}\sin(t)-2e^{-\pi}\cos(t)+e^{-\pi}\sin(t)\right).$$

5. (5 total points)

(a) (3 points) A ball with mass m is dropped (i. e. with zero initial velocity) from the roof of a building. Assume that the force due to air resistance is proportional to the square of its velocity where the velocity is measured in m/s. Determine the velocity of the ball for all sufficiently small times $t > 0$ (i. e. you do not need to worry about what happens after the ball hits the ground). You need to mention if you take the downward direction to be the positive or negative direction; either choice is fine.

We'll assume that downward is the positive direction. By Newton's second law of motion, the velocity of the ball satisfies

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = F = mg - kv^2, \quad v(0) = 0.$$

This equation is separable, so

$$\frac{m}{mg-kv^2}\mathrm{d}v = \mathrm{d}t,$$

and using the table of integrals,

$$t + C = \int \mathrm{d}t = \int \frac{m}{mg-kv^2}\mathrm{d}v = -\frac{m}{k}\int\frac{1}{v^2-\frac{mg}{k}}\mathrm{d}v = -\frac{m}{k}\frac{1}{2\sqrt{\frac{mg}{k}}}\ln\left|\frac{v-\sqrt{\frac{mg}{k}}}{v+\sqrt{\frac{mg}{k}}}\right|$$

Using the initial condition yields

$$C = -\frac{m}{k}\frac{1}{2\sqrt{\frac{mg}{k}}}\ln|-1| = 0.$$

Thus

$$t = -\frac{m}{k}\frac{1}{2\sqrt{\frac{mg}{k}}}\ln\left|\frac{v-\sqrt{\frac{mg}{k}}}{v+\sqrt{\frac{mg}{k}}}\right| = -\frac{1}{2}\sqrt{\frac{m}{kg}}\ln\left|\frac{\sqrt{k}v-\sqrt{mg}}{\sqrt{k}v+\sqrt{mg}}\right| = \frac{1}{2}\sqrt{\frac{m}{kg}}\ln\left|\frac{\sqrt{k}v+\sqrt{mg}}{\sqrt{k}v-\sqrt{mg}}\right|.$$

Finally, we solve for v:

$$\begin{aligned} t = \frac{1}{2}\sqrt{\frac{m}{kg}}\ln\left(\frac{\sqrt{mg}+\sqrt{k}v}{\sqrt{mg}-\sqrt{k}v}\right) \quad &\Rightarrow \quad e^{2\sqrt{\frac{k}{m}g}t} = \frac{mg+kv}{mg-kv} \\ &\Rightarrow \quad \left(\sqrt{mg}-\sqrt{k}v\right)e^{2\sqrt{\frac{k}{m}g}t} = \sqrt{mg}+\sqrt{k}v \\ &\Rightarrow \quad -\sqrt{k}ve^{2\sqrt{\frac{k}{m}g}t} - \sqrt{k}v = -\sqrt{mg}e^{2\sqrt{\frac{k}{m}g}t} + \sqrt{mg} \\ &\Rightarrow \quad v = \sqrt{\frac{m}{k}g}\,\frac{e^{2\sqrt{\frac{k}{m}g}t}-1}{e^{2\sqrt{\frac{k}{m}g}t}+1}. \end{aligned}$$

(b) (1 point) Assume the situation in part (a) except that the ball is dropped from an airplane. Since the ball is dropped from a considerable height, we need to assume that the resistance force depends not only on air resistance, but also on air pressure. Assume that the resistance is proportional to the product of air resistance and air pressure. Also, assume that the air pressure varies exponentially with the altitude. You can assume that the force by gravity is independent of the altitude of the ball. Set up a differential equation which describes the motion. *Note: A quantity x varies exponentially if it is proportional to* e^{-ax} *for some* $a > 0$.

The motion is described by

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - kv^2e^{-as}.$$

To get a differential equation, we need to write it in terms of the altitude s, so

$$m\frac{\mathrm{d}^2s}{\mathrm{d}t^2} = mg - k\frac{\mathrm{d}s}{\mathrm{d}t}^2 e^{-as}.$$

(c) (1 point) Show that the velocity of the object satisfies a first-order differential equation. *Hint: Proceed to solve the differential equation you set up in (b). The result will pop out mid-way.*

This differential equation is of the type $s'' = f(s,s')$, so we can apply the change of variables $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}s}v$. Then

$$m\frac{\mathrm{d}v}{\mathrm{d}s}v = mg - kv^2e^{-as}.$$

That is the required first order differential equation.