1. (5 total points)

   (a) (4 points) Suppose the position of some object at time t is described by the following initial value problem:

   $$y' = \frac{1}{ty}, \quad t > 0, \quad y(1) = 1.$$

   Determine the position of the object at time t.

   The equation is separable, so we need to compute

   $$\int y(t)y'(t)\mathrm{d}t = \int \frac{1}{t}\mathrm{d}t.$$

   For the right hand side,

   $$\int \frac{1}{t}\mathrm{d}t = \ln t + C,$$

   (we can write $\ln t$ since we assumed $t > 0$) and for the left hand side we pick the usual substitution $u = y(t)$, so $\mathrm{d}u = y'(t)\mathrm{d}t$, and

   $$\int y(t)y'(t)\mathrm{d}t = \int u\mathrm{d}u = \frac{1}{2}u^2 = \frac{1}{2}y^2(t).$$

   Thus

   $$\frac{1}{2}y^2(t) = \ln t + C.$$

   Now, solve for $y(t)$,

   $$\frac{1}{2}y^2(t) = \ln t + C \Rightarrow y^2(t) = 2\ln(t) + C \Rightarrow y(t) = \pm\sqrt{2\ln t + C}.$$

   To determine C and the sign in front of the square root (called the branch), use the initial condition,

   $$1 = y(1) = \pm\sqrt{2\ln 1 + C} = \pm\sqrt{C},$$

   so clearly we have to pick the plus sign, and $C = 1$. Thus the solution is

   $$y(t) = \sqrt{2\ln t + 1}.$$

   (b) (1 point) For which times t is the solution defined?

   We have to guarantee two things: First, the argument of the logarithm has to be non-negative. It is, since we assumed that $t > 0$. Second, we need to make sure that we only evaluate the square-root at non-negative numbers, i. e. $2\ln t + 1 \geq 0$. Now,

   $$2\ln t + 1 \geq 0 \Leftrightarrow 2\ln(t) \geq -1 \Leftrightarrow \ln(t) \geq -\frac{1}{2} \Leftrightarrow t \geq e^{-\frac{1}{2}}.$$

   Thus the solution is defined for $t \geq e^{-\frac{1}{2}}$.

2. (4 points) Determine explicitly all the solutions to the differential equation

   $$(1+t^2)y' + y = 1.$$

This equation is linear. First, divide by $1+t^2$ to bring it into the standard form,

$$y' + \frac{1}{1+t^2}y = \frac{1}{1+t^2}.$$

and multiply by some integrating factor μ, so

$$\mu y' + \mu\frac{1}{1+t^2}y = \mu\frac{1}{1+t^2}.$$

To use the product rule (in reverse) on the left hand side, we need μ to satisfy

$$\mu'(t) = \mu(t)\frac{1}{1+t^2}.$$

Using the list of integrals,

$$\log|\mu(t)| = \int \frac{\mu'(t)}{\mu(t)}\mathrm{d}t = \int \frac{1}{1+t^2}\mathrm{d}t = \arctan(t)$$

so one integrating factor is

$$\mu(t) = e^{\arctan(t)}.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(ye^{\arctan(t)}\right) = \frac{1}{1+t^2}e^{\arctan(t)}.$$

Integrating both sides yields

$$ye^{\arctan(t)} = \int \frac{1}{1+t^2}e^{\arctan(t)}\mathrm{d}t.$$

We just saw above that $\arctan(t)$ is the antiderivative of $\frac{1}{1+t^2}$ which calls for the substitution $u = \arctan(t)$, so that $\mathrm{d}u = \frac{1}{1+t^2}\mathrm{d}t$. Thus

$$ye^{\arctan(t)} = \int \frac{1}{1+t^2}e^{\arctan(t)}\mathrm{d}t = \int e^u \mathrm{d}u = e^u + C = e^{\arctan(t)} + C.$$

Finally, solve for y to obtain

$$y = 1 + Ce^{-\arctan(t)}.$$

3. (6 points) Initially, a tank contains 6 gal of water containing 1 lb of salt. There is water flowing into the tank through two pipes: Water containing salt is entering the tank through the first pipe at rate of 2 gal/min. Several measurements indicate that the amount of salt contained in one gallon of the incoming water is $e^{-\frac{3}{2}t}$ lb at time t. One gallon of fresh water per minute is entering the tank through the second pipe. Finally, the well-stirred mixture is draining the tank at a rate of 3 gal/min.

   Determine the amount of salt at any time $t \geq 0$.

Let $Q(t)$ be the amount of salt at time t in lb. Then $Q(0) = 1$, and the time derivative satisfies

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \text{rate in} - \text{rate out} = e^{-\frac{3}{2}t}\frac{\text{lb}}{\text{gal}}\cdot 2\frac{\text{gal}}{\text{min}} + 0\frac{\text{lb}}{\text{gal}}\cdot 1\frac{\text{gal}}{\text{min}} - \frac{Q(t)}{6}\frac{\text{lb}}{\text{gal}}\cdot 3\frac{\text{gal}}{\text{min}} = 2e^{-\frac{3}{2}t} - \frac{1}{2}Q\frac{\text{lb}}{\text{min}}$$

We multiply through be the integrating factor $\mu(t)$, so

$$\mu(t)\frac{\mathrm{d}Q}{\mathrm{d}t}+\mu(t)\frac{1}{2}Q=2\mu(t)e^{-\frac{3}{2}t},$$

and we therefore need

$$\mu'(t)=\frac{1}{2}\mu(t)\Rightarrow\mu(t)=e^{\frac{1}{2}t}.$$

Now,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Qe^{\frac{1}{2}t}\right)=2e^{\frac{1}{2}t}e^{-\frac{3}{2}t}=2e^{-t}$$

Integrating both sides yields

$$Qe^{\frac{1}{2}t}=-2e^{-t}+C$$

and thus

$$Q=-2e^{-\frac{3}{2}t}+Ce^{-\frac{1}{2}t}$$

Use the initial condition to deduce

$$1=Q(0)=-2+C$$

so $C=3$, and hence the amount of salt at any given time $t\geq 0$ is

$$Q(t)=-2e^{-\frac{3}{2}t}+3e^{-\frac{1}{2}t}$$

4. (3 total points) Suppose that fish are harvested a constant rate E from the total population. Then we have to modify the logistic equation as follows:

$$\frac{\mathrm{d}P}{\mathrm{d}t}=r\left(1-\frac{P}{K}\right)P-EP$$

where P is a function of time and represents the number of the fish at time t, K and r are positive constants, and $E\geq 0$ is a nonnegative constant.

(a) (5 points) Assume that $E<r$. Determine and classify all the equilibrium solutions to this equation.

Note that the right hand side is a quadratic polynomial, so the best approach is to factor:

$$r\left(1-\frac{P}{K}\right)P-EP=P\left(r-r\frac{P}{K}-E\right)=-\frac{r}{K}P\left(P+K\left(\frac{E}{r}-1\right)\right)=\left(P-K\left(1-\frac{E}{r}\right)\right)$$

Since equilibrium solutions satisfy $\frac{\mathrm{d}P}{\mathrm{d}t}=0$, we see that $P=0$ and $P=K\left(1-\frac{E}{r}\right)$ are the equilibrium solutions. Note that the latter is indeed an equilibrium solution since $E<r$, so $K\left(1-\frac{E}{r}\right)>0$.

To determine where P is increasing and decreasing, we look at both factors: $-\frac{r}{K}P$ is always negative, and $\left(P-K\left(1-\frac{E}{r}\right)\right)$ is negative if $P<K\left(1-\frac{E}{r}\right)$ and positive if $P>K\left(1-\frac{E}{r}\right)$. Therefore, $\frac{\mathrm{d}P}{\mathrm{d}t}>0$ and P is increasing if $P<K\left(1-\frac{E}{r}\right)$; $\frac{\mathrm{d}P}{\mathrm{d}t}<0$ and P is decreasing if $P>K\left(1-\frac{E}{r}\right)$. We conclude that $P=0$ is unstable, and $P=K\left(1-\frac{E}{r}\right)$ is asymptotically stable.

(b) (1 point) How does the number and classification of the equilibrium solutions change if we assume $E > r$?

If $E > r$, then $K\left(1 - \frac{E}{r}\right) < 0$, so this is not an equilibrium solution anymore. Moreover, $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$ and P is always decreasing. We conclude that $P = 0$ is asymptotically stable.

5. (3 points) A kid places a skyrocket in a bottle, burns the fuse, and runs away. After 5 seconds, the skyrocket launches straight into the sky. The rocket weights 0.4 kg, including 0.2 kg of propellant. For that rocket, the force due to air resistance has been measured to be $|v|/60$ N. After ignition, the propellant burns down at a constant rate of 0.04 kg/s and thereby exerts a constant force of 20 N. Once all the propellant is burnt, the skyrocket explodes.

   Write down, but do NOT solve, an initial value problem for the velocity of the skyrocket up to the time where it explodes, as well as the time interval for which the differential equation is valid.

Using Newton's law of motion,

$$mv' = ma = F = -mg - \frac{v}{60} + 20$$

However, this time the the mass depends on time t. Indeed, we start with 0.4 kg and loose 0.04 kg per second for five seconds, so $m(t) = 0.4 - 0.04t$. Thus

$$(0.4 - 0.04t)\,v' = ma = F = -\,(0.4 - 0.04t)\,g - \frac{v}{60} + 20,$$

and the equation is valid for 5 seconds after the skyrocket is launched. For instance, if we take the time when the rocket launches to be time zero, $0 \le t \le 5$. In that case, the initial condition is $v(0) = 0$.