1. (10 total points) Find the expicit general solutions to the following first-order differential equations. Your answer should be a function in the form $y = g(x, C)$, where C is an integration constant parameterizing the family of solutions to the DE.

   (a) (5 points) $\frac{dy}{dx} - 2xy - x = 0$

   This problem is linear, so we put it in standard form: $y' + f(x) \cdot y = g(x)$. Hence

$$\frac{dy}{dx} + (-2x)y = x$$

   Our integrating factor is thus $\mu(x) = e^{\int f(x)\, dx}$, so

$$\mu(x) = e^{\int -2x\, dx} = e^{-x^2}$$

   The general solution to the DE is then

$$y(x) = \frac{1}{\mu(x)} \left(\int \mu(x) g(x)\, dx + C \right),$$

   so here we have

$$\begin{aligned} y(x) &= e^{x^2} \left(\int e^{-x^2} \cdot x\, dx + C \right) \\ &= e^{x^2} \left(-\frac{1}{2} e^{-x^2} + C \right) \\ &= -\frac{1}{2} + Ce^{x^2}. \end{aligned}$$

   So the general solution to the DE is

$$y = -\frac{1}{2} + Ce^{x^2}.$$

(b) (5 points) $\frac{dy}{dx} = \frac{y^2-y}{x^2+1}$

This equations is separable. Putting all the y stuff on the left and all the x stuff on the right yields

$$\frac{1}{y^2-y}\,dy = \frac{1}{x^2+1}\,dx$$

The differentials are equal, so their antiderivatives must agree up to a constant. On the right hand side we have

$$\int \frac{1}{x^2+1}\,dx = \arctan(x) + C_1,$$

where C_1 is the unknown constant. The integral on the left can be done using partial fractions: write $\frac{1}{y^2-y} = \frac{A}{y-1} + \frac{B}{y}$, so after clearing denominators we get

$$1 = Ay + B(y-1).$$

Substituting $y = 1$ and $y = 0$ yield $A = 1$ and $B = -1$ respectively. Hence

$$\begin{aligned}\int \frac{1}{y^2-y}\,dy &= \int \frac{1}{y-1} - \frac{1}{y}\,dy \\ &= \ln|y-1| - \ln|y| \\ &= \ln\left|\frac{y-1}{y}\right| \\ &= \ln\left|1 - \frac{1}{y}\right|.\end{aligned}$$

Hence we have

$$\ln\left|1 - \frac{1}{y}\right| = \arctan(x) + C_1.$$

Exponentiating both sides and letting $C = \pm e^{C_1}$ we get

$$1 - \frac{1}{y} = Ce^{\arctan(x)}.$$

Thus, when we solve for y we get the general solution

$$y = \frac{1}{1 - Ce^{\arctan(x)}}.$$

2. (10 total points) Solve the following initial value problems. Your answer should be in the form $y = g(t)$, where there is no undetermined constant in g.

   (a) (5 points) $\frac{dy}{dt} = \frac{1-3ty}{t^2}$, $\quad y(1) = 0$.

   This equation is linear. In standard form it is

$$\frac{dy}{dt} + \frac{3}{t} = \frac{1}{t^2}.$$

   The integrating factor is thus

$$\mu(t) = e^{\int \frac{3}{t}\,dt} = e^{3\ln|t|} = |t|^3.$$

   The initial condition is at $t = 1$, so we can (for now at least) assume $t > 0$; we thus have that $\mu(t) = t^3$. The general solution is therefore

$$\begin{aligned} y(t) &= t^{-3}\left(\int t^3 \cdot t^{-2}\,dt + C\right) \\ &= t^{-3}\left(\frac{1}{2}t^2 + C\right) \\ &= \frac{1}{2t} + \frac{C}{t^3}. \end{aligned}$$

   Plugging in the initial condition $(t,y) = (1,0)$ has

$$0 = \frac{1}{2} + C,$$

   so $C = -\frac{1}{2}$. The solution to the IVP is thus

$$y = \frac{1}{2t} - \frac{1}{2t^3} = \frac{(t-1)(t+1)}{2t^3}.$$

(b) (5 points) $\frac{dy}{dt} = e^{2t-3y}, \quad y(1) = 2.$

This equation is separable. Separating the variables we get

$$e^{3y}\,dy = e^{2t}\,dx.$$

Antidifferentiating both dies gives us

$$\frac{1}{3}e^{3y} = \frac{1}{2}e^{2t} + C,$$

or, after solving for y,

$$y = \frac{1}{3}\ln\left(\frac{3}{2}e^{2t} + C\right),$$

where we've absorbed the factor of 3 into the C. Substituting in the initial conditions $y(1) = 2$ gives us

$$2 = \frac{1}{3}\ln\left(\frac{3e^2}{2} + C\right),$$

so $C = e^6 - \frac{3}{2}e^2$. Hence the solution to the IVP is

$$y = \frac{1}{3}\ln\left(\frac{3}{2}e^{2t} + e^6 - \frac{3}{2}e^2\right).$$

3. (10 points) The differential equation

$$\frac{dy}{dx} = \frac{2y+3x}{2y+2x}$$

is neither linear nor separable. Use the homogenizing substitution $v = \frac{y}{x}$ to find the solution to the DE with the initial condition $y(1) = 0$. Along with your answer state explicitly on what interval the solution is defined.

[Note: When I cooked up this question and checked to see it was midterm-level difficulty I made an error while solving it; one of the integrals you end up doing is thus quite a bit messier than I wanted it to be. As a result this problem is far more finicky than what you would see on an exam. Nevertheless it is at least implicitly solvable, so I reproduce the full solution below for good measure.]

First, note that we can write the right hand side of the equation, if we divide both the top and bottom by $\frac{1}{x}$, as

$$\frac{2\frac{y}{x}+3}{2\frac{y}{x}+2} = \frac{2v+3}{2v+2}.$$

However, to convert this into a DE involving v and x only we must still relate $\frac{dv}{dx}$ to $\frac{dy}{dx}$. To do this, note that if $v = \frac{y}{x}$ then $y = vx$, and we can take implicit derivatives (+ product rule) of this last equation to get

$$\frac{dy}{dx} = x\frac{dv}{dx} + v.$$

Hence we have the differential equation

$$x\frac{dv}{dx} + v = \frac{2v+3}{2v+2}.$$

We can also convert our initial conditions: $y(1) = 0 \Rightarrow v(1) = 0/1 = 0$.

The differential equation is, after some algebraic mangling, separable. After we subtract v from both sides we get

$$\begin{aligned} x\frac{dv}{dx} &= \frac{2v+3}{2v+2} - v \\ &= \frac{2v+3}{2v+2} - \frac{v(2v+2)}{2v+2} \\ &= \frac{3-2v^2}{2v+2}. \end{aligned}$$

Hence

$$\frac{2v+2}{3-2v^2}\,dv = \frac{1}{x}\,dx.$$

The right hand side integrates to $\ln(x) + C$; we may assume $x > 0$ in this case, since the initial condition is given at $x = 1$.

[Solution continued overleaf]

The left hand integral can be done via integration by parts. Note that

$$\frac{2v+2}{3-2v^2} = \frac{4v+4}{6-4v^2} = \frac{4v+4}{(\sqrt{6}-2v)(\sqrt{6}+2v)}.$$

Setting the last expression to $\frac{A}{\sqrt{6}-2v} + \frac{B}{\sqrt{6}+2v}$ and solving for A and B in the usual way gives us that

$$\frac{2v+2}{3-2v^2} = \left(\frac{\sqrt{6}}{3}+1\right)\cdot\frac{1}{\sqrt{6}-2v} + \left(\frac{\sqrt{6}}{3}-1\right)\cdot\frac{1}{\sqrt{6}+2v};$$

Hence the integral on the left becomes

$$\begin{aligned}\int \frac{2v+2}{3-2v^2}\,dv &= \frac{1}{2}\left(\frac{\sqrt{6}}{3}-1\right)\ln\left|\sqrt{6}+2v\right| - \frac{1}{2}\left(\frac{\sqrt{6}}{3}+1\right)\ln\left|\sqrt{6}-2v\right| \\ &= \frac{1}{6}\left[(\sqrt{6}-3)\ln\left(\sqrt{6}+2v\right) - (\sqrt{6}+3)\ln\left(\sqrt{6}-2v\right)\right].\end{aligned}$$

We may drop the absolute value signs, as close to the initial condition point $x=1, y=0$ we have $v=y/x$ close to zero, so both $\sqrt{6}+2v$ and $\sqrt{6}-2v$ are positive.

We thus have that

$$\frac{1}{6}\left[(\sqrt{6}-3)\ln\left(\sqrt{6}+2v\right) - (\sqrt{6}+3)\ln\left(\sqrt{6}-2v\right)\right] = \ln(x)+C,$$

so, after multiplying by 6 we have

$$(\sqrt{6}-3)\ln\left(\sqrt{6}+2v\right) - (\sqrt{6}+3)\ln\left(\sqrt{6}-2v\right) = 6\ln(x)+C$$

(absorbing the 6 into the C). We can now take exponentiate both sides to get

$$(\sqrt{6}+2v)^{\sqrt{6}-3}(\sqrt{6}-2v)^{-\sqrt{6}-3} = Ax^6.$$

Applying the IC $x=1, v=0$ yields, after simplification, that $A = \frac{1}{216}$.

Thus, when simplified, we get the implicit equation

$$\left(\frac{\sqrt{6}+2v}{\sqrt{6}-2v}\right)^{\sqrt{6}}\cdot(6-4v)^{-3} = \frac{x^6}{216}.$$

We can then back substitute $v=\frac{y}{x}$ to recover an implicit equation in our original variables:

$$\left(\frac{\sqrt{6}x+2y}{\sqrt{6}x-2y}\right)^{\sqrt{6}}\cdot\left(6-4\cdot\frac{y}{x}\right)^{-3} = \frac{x^6}{216}.$$

That's about as simplified as we can get in this case, so we let it lie here. Please remember that this question is far trickier than what you'll see in an exam.

4. (10 total points) A reservoir on a farm initially contains 10000 liters of water, in which 200kg nitrate fertilizer is dissolved. The owner of the reservoir decides the amount of dissolved nitrate needs to be increased, so starts pumping in a 1kg nitrate:1 liter water solution at a rate of 100 l/min. However, the reservoir simultaneously develops a leak, and starts draining at a rate of 200 l/min.

   (a) (7 points) Assuming the solution remains perfectly mixed at all times, find the mass of nitrate in the reservoir at time t.

   Let $y(t)$ be the mass in kg of nitrate in the reservoir at time t, where t is in minutes, and let $V(t)$ be the volume of water in liters in the reservoir at time t. We collect the following facts from the problem description:

   - $V(0) = 10000$
   - $\frac{dV}{dt} = -100$
   - $y(0) = 200$
   - The rate at which nitrate is entering the tank is 100 l/min $\times$1kg/l $=$ 100kg/min
   - We assume the solution in the reservoir is well-mixed at all times, so the rate at which nitrate is exiting the tank is the outflow rate times the nitrate concentration, i.e. $200 \times \frac{y(t)}{V(t)}$.

   Now we know the rate at which the tank is draining is linear, i.e. combining the first two bullet points gives us the volume in the tank at time t:

$$V(t) = 10000 - 100t.$$

   Note that this is valid for $0 \leq t \leq 100$; at that point the tank has emptied, so both V and y are zero from then on.

   Finally, this is a mixing problem: we know that $\frac{dy}{dt}$ = rate in $-$ rate out. Thus, combining all the info above we get the IVP

$$\frac{dy}{dt} = 100 - 200 \cdot \frac{y}{10000 - 100t} \quad y(0) = 200.$$

   The differential equation is linear; in standard form after simplifying it is

$$\frac{dy}{dt} + \frac{2}{100-t}y = 100.$$

   To solve it we determine the integrating factor to be $\mu(t) = e^{\int \frac{2}{100-t}} = (100-t)^{-2}$; since we are only considering the case $0 < t < 100$ we don't need the absolute value signs when we integrate $\frac{2}{100-t}$. The general solution is then

$$\begin{aligned} y(t) &= (100-t)^2 \left(\int (100-t)^{-2} \cdot 100\, dt + C \right) \\ &= (100-t)^2 \left(100(100-t)^{-1} + C \right) \\ &= 100(100-t) + C(100-t)^2. \end{aligned}$$

   We plug in the initial condition $y(0) = 200$ to get $200 = 100^2 + C(100)^2$, so $C = -\frac{49}{50}$. The solution to the differential equation is thus

$$y(t) = -\frac{49}{50}(100-t)^2 + 100(100-t) = -\frac{49}{50}t^2 + 96t + 200.$$

   Note that this is valid only for $0 \leq t \leq 100$; after this the reservoir has emptied, so $y = 0$.

(b) (3 points) What is the maximum amount of nitrate in the reservoir, and when does it occur?

The solution to the IVP is

$$y(t) = -\frac{49}{50}t^2 + 96t + 200.$$

This is a quadratic function with a negative coefficient in front of the t^2 term (a "sad quadratic"), so we know its turning point is a global max. The time at which it occurs is

$$t = \frac{-96}{2 \cdot \frac{49}{50}} = \frac{2400}{49} \simeq 48.9796,$$

and the corresponding y-value is thus

$$y = -\frac{49}{50}\left(\frac{2400}{49}\right)^2 + 96\left(\frac{2400}{49}\right) + 200 = \frac{125000}{49} \simeq 2551.0204.$$

In other words, to the nearest kilogram and minute respectively the maximum amount of nitrate in the tank is about 2551kg, occurring at around the 49 minute mark.

5. (10 total points) We've seen in class and the homework a model for objects under free fall in which we assume the drag force acting on an object is proportional to its velocity. In reality a more accurate model is one in which drag force is proportional to the velocity *squared*. As such, a differential equation to model the velocity $v(t)$ of a falling object near the earth's surface is

$$\frac{dv}{dt} = -g + kv^2,$$

where g is acceleration due to gravity, and the proportionality constant k depends on the mass and aerodynamics of the object being dropped as well as atmospheric conditions.

(a) (5 points) For skydivers in freefall under standard atmospheric conditions, it's known that k is approximately $2.73 \times 10^{-3}\,m^{-1}$ (i.e. when working in metric k has units of inverse meters). Using this value and taking $g = 9.81\,ms^{-2}$, compute the terminal (limiting) velocity of such a skydiver. You may use decimal approximations in your final answer (but keep at least 4 digits precision at all points).

This differential equation is separable, so a perfectly valid method to answering this question is to solve the DE and take the limit as $t \to \infty$ of the solution. However, the equation is autonomous, so it is much quicker to use the qualitative techniques available for analyzing autonomous equations. Note that the right hand side of the DE is a quadratic in v, and can be written as

$$g\left(\frac{k}{g}v^2 - 1\right) = g\left(\sqrt{\frac{k}{g}}v - 1\right)\left(\sqrt{\frac{k}{g}}v + 1\right).$$

In other words $\frac{dv}{dt} = 0$ when $v = \pm\sqrt{\frac{g}{k}}$. Both of these correspond to equilibrium solutions to the DE. However, since we are investigating a system where velocity is negative (looking at the DE, we see we have defined down as negative), we are interested in the negative solution.

Furthermore, this is a quadratic with a positive coefficient in front of the v^2 term, so we know that the slope at the root on the left is negative; in other words, the equilibrium solution $v = -\sqrt{\frac{g}{k}}$ is a stable solution; other solutions will thus tend to $v = -\sqrt{\frac{g}{k}}$ as $t \to \infty$. We conclude that this must be the limiting or terminal velocity of the skydiver.

Plugging in numbers we have

$$v = -\sqrt{\frac{g}{k}} = \sqrt{\frac{9.81}{2.73 \times 10^{-3}}} \simeq 59.9450.$$

That is, terminal velocity for a human skydiver is about 60 ms^{-1}.

(b) (5 points) Use Euler's method with a step size of $h = 0.5$ to estimate a skydiver's downward velocity 1 second after jumping from a plane. You may use decimal approximations in your final answer (but keep at least 4 digits precision at all points).

Recall that Euler's method for the IVP $\frac{dv}{dt} = f(t,v),\ \ v(t_0) = v_0$ with step size h is given by the scheme

- Set t_0 and v_0 to be the given initial conditions
- for $n \geq 0$ set $t_{n+1} = t_0 + h$ and $v_{n+1} = v_0 + h \cdot f(t_n, v_n)$.

We have $h = 0.5$, $t_0 = 0$, $v_0 = 0$ (by assumption) and $f(t,v) = -9.81 + 2.73 \cdot 10^{-3} \cdot v^2$. Hence

$$v_1 = 0 + 0.5 * (-9.81 + 2.73 \cdot 10^{-3} \cdot 0^2) = -4.905$$

and

$$v_1 = -4.905 + 0.5 * (-9.81 + 2.73 \cdot 10^{-3} \cdot (-4.905)^2) \simeq -9.7772.$$

Thus Euler's method with $h = 0.5$ predicts that a skydiver will have velocity about $-9.7772\, ms^{-1}$ after 1 second of free fall.

(Note that this is very close to the velocity we'd get without any air resistance, namely $-9.81\, ms^{-1}$.)