3.5: Non-homogeneous Constant Coefficient Second Order (Undetermined Coefficients)

Given $ay'' + by' + cy = g(t)$, $y(t_0) = y_0$ and $y'(t_0) = y_0'$.

Step 1: Find the general solution of the homogeneous equation $ay'' + by' + cy = 0$.
(Write and solve the characteristic equation, then use methods from 3.1, 3.3, and 3.4).
At this point, you'll have two independent solutions to the homogeneous equation: $y_1(t)$ and $y_2(t)$.

Step 2: From the table below, identify the likely form of the answer of a **particular solution**, $Y(t)$, to $ay'' + by' + cy = g(t)$. *Table of Particular Solution Forms*

$g(t)$	e^{rt}	$\sin(\omega t)$ or $\cos(\omega t)$	C	t	t^2	t^3
$Y(t)$	Ae^{rt}	$A\cos(\omega t) + B\sin(\omega t)$	A	$At + B$	$At^2 + Bt + C$	$At^3 + Bt^2 + Ct + D$

First some notes on the use of this table:

- If $g(t)$ is a sum/difference of these types, then so is $Y(t)$.
  For example, if $g(t) = e^{4t} + \sin(5t)$, then try $Y(t) = Ae^{4t} + B\cos(5t) + C\sin(5t)$.

- If $g(t)$ is a product of these types, then so is $Y(t)$, but you don't need to have an extra coefficient factored throughout.
  For example, if $g(t) = t^2e^{5t}$, then try $Y(t) = (At^2 + Bt + C)e^{5t}$. You don't need to write $(At^2 + Bt + C)De^{5t}$ because the D could be multiplied through to give $(ADt^2 + BDt + CD)e^{5t}$ which is of the first form already and the D is not needed. Here are tables of the typical products:

$g(t)$	te^{rt}	t^2e^{rt}	$t\sin(\omega t)$ or $t\cos(\omega t)$
$Y(t)$	$(At + B)e^{rt}$	$(At^2 + Bt + C)e^{rt}$	$(At + B)\cos(\omega t) + (Ct + D)\sin(\omega t)$

$g(t)$	$e^{rt}\cos(\omega t)$ or $e^{rt}\sin(\omega t)$	$te^{rt}\cos(\omega t)$ or $te^{rt}\sin(\omega t)$
$Y(t)$	$Ae^{rt}\cos(\omega t) + Be^{rt}\sin(\omega t)$	$(At + B)e^{rt}\cos(\omega t) + (Ct + D)e^{rt}\sin(\omega t)$

- **Important:** How to adjust for homogeneous solutions
  Consider a particular term of $g(t)$. If the table suggests you use the form $Y(t)$ for this term, but $Y(t)$ **contains a constant multiple of a homogeneous solution, then you need to multiply by** t (and if that still constains a homogeneous solution, then multiple by t again).

  For example, $g(t) = te^{2t}$, then you would initially guess the form $Y(t) = (At + B)e^{2t}$. But if the homogeneous solutions are $y_1(t) = e^{2t}$ and $y_2(t) = e^{5t}$, then Be^{2t} is a multiple of a homogeneous solution. So you use the form: $Y(t) = t(At + B)e^{2t} = (At^2 + Bt)e^{2t}$.

  For another example, if $g(t) = te^{7t}$, then you would initially guess the form $Y(t) = (At + B)e^{7t}$. But if the homogeneous solutions are $y_1(t) = e^{7t}$ and $y_2(t) = te^{7t}$, then Be^{7t} AND Ate^{7t} are both multiples of a homogeneous solution. So you use the form: $Y(t) = t^2(At + B)e^{7t} = (At^3 + Bt^2)e^{7t}$

Step 3: Compute $Y'(t)$ and $Y''(t)$. Substitute $Y(t)$, $Y'(t)$ and $Y''(t)$ into $ay'' + by' + cy = g(t)$.

Step 4: Solve for the coefficients and write your general solution:

$$y(t) = c_1y_1(t) + c_2y_2(t) + Y(t)$$

Step 5: Use the initial conditions and solve for c_1 and c_2.

Here are some problems to practice identifying the correct form.
In each line, you are given $g(t)$ as well as independent homogeneous solutions $y_1(t)$, and $y_2(t)$.
Give the form of the particular solution, $Y(t)$ (solutions below).

1.	$ay'' + by' + cy = e^{2t}$	$y_1(t) = \cos(t)$	$y_2(t) = \sin(t)$
2.	$ay'' + by' + cy = \cos(3t)$	$y_1(t) = e^{3t}$	$y_2(t) = e^{-t}$
3.	$ay'' + by' + cy = e^{4t}$	$y_1(t) = e^{4t}$	$y_2(t) = e^{-2t}$
4.	$ay'' + by' + cy = t$	$y_1(t) = e^{6t}$	$y_2(t) = te^{6t}$
5.	$ay'' + by' + cy = e^{3t}$	$y_1(t) = e^{3t}$	$y_2(t) = te^{3t}$
6.	$ay'' + by' + cy = e^{t}\sin(5t)$	$y_1(t) = e^{-t}$	$y_2(t) = e^{6t}$
7.	$ay'' + by' + cy = \sin(t) + t$	$y_1(t) = e^{-2t}\cos(4t)$	$y_2(t) = e^{-2t}\sin(4t)$
8.	$ay'' + by' + cy = \cos(2t)$	$y_1(t) = \cos(2t)$	$y_2(t) = \sin(2t)$
9.	$ay'' + by' + cy = 5 + e^{2t}$	$y_1(t) = e^{3t}$	$y_2(t) = e^{-6t}$
10.	$ay'' + by' + cy = te^{2t}\cos(5t)$	$y_1(t) = e^{t}$	$y_2(t) = te^{t}$
11.	$ay'' + by' + cy = te^{7t}$	$y_1(t) = e^{7t}$	$y_2(t) = e^{-5t}$
12.	$ay'' + by' + cy = t^2e^{7t}$	$y_1(t) = e^{7t}$	$y_2(t) = te^{-4t}$
13.	$ay'' + by' + cy = te^{8t}$	$y_1(t) = e^{8t}$	$y_2(t) = te^{8t}$
14.	$ay'' + by' + cy = t^2e^{-t}\sin(9t)$	$y_1(t) = \cos(9t)$	$y_2(t) = \sin(9t)$
15.	$ay'' + by' + cy = t^2 - 5 + te^{t}$	$y_1(t) = e^{t}\cos(5t)$	$y_2(t) = e^{t}\sin(5t)$

Solutions

1. $Y(t) = Ae^{2t}$.
2. $Y(t) = A\cos(3t) + B\sin(3t)$.
3. $Y(t) = Ate^{4t}$.
4. $Y(t) = At + B$.
5. $Y(t) = At^2e^{3t}$.
6. $Y(t) = Ae^{t}\cos(5t) + Be^{t}\sin(5t)$.
7. $Y(t) = A\cos(t) + B\sin(t) + Ct + D$.
8. $Y(t) = At\cos(2t) + Bt\sin(2t)$.
9. $Y(t) = A + Be^{2t}$.
10. $Y(t) = (At + B)e^{2t}\cos(5t) + (Ct + D)e^{2t}\sin(5t)$.
11. $Y(t) = (At^2 + Bt)e^{7t}$.
12. $Y(t) = (At^3 + Bt^2 + Ct)e^{7t}$.
13. $Y(t) = (At^3 + Bt^2)e^{8t}$.
14. $Y(t) = (At^2 + Bt + C)e^{-t}\cos(9t) + (At^2 + Bt + C)e^{-t}\sin(9t)$.
15. $Y(t) = At^2 + Bt + C + (Dt + E)e^{t}$.

Examples:

1. Give the general solution to $y'' + 10y' + 21y = 5e^{2t}$.

   Solution:

   (a) *Solve Homogeneous*:
   The equation $r^2 + 10r + 21 = (r+3)(r+7) = 0$ has the roots $r_1 = -3$ and $r_2 = -7$.
   So $y_1(t) = e^{-3t}$ and $y_2(t) = e^{-7t}$

   (b) *Particular Solution Form*:
   $Y(t) = Ae^{2t}$

   (c) *Substitute*:
   $Y'(t) = 2Ae^{2t}$ and $Y''(t) = 4Ae^{2t}$. Substituting gives
   $4Ae^{2t} + 10(2Ae^{2t}) + 21(Ae^{2t}) = 5e^{2t} \Rightarrow 45Ae^{2t} = 5e^{2t}$. Thus, $A = \frac{5}{45} = \frac{1}{9}$.

   (d) *General Solution*:
   $y(t) = c_1e^{-3t} + c_2e^{-7t} + \frac{1}{9}e^{2t}$.

2. Give the general solution to $y'' - 2y' + y = 6t$.

   Solution:

   (a) *Solve Homogeneous*:
   The equation $r^2 - 2r + 1 = (r-1)^2 = 0$ has the one root $r = 1$.
   So $y_1(t) = e^t$ and $y_2(t) = te^t$.

   (b) *Particular Solution Form*:
   $Y(t) = At + B$

   (c) *Substitute*:
   $Y'(t) = A$ and $Y''(t) = 0$. Substituting gives
   $(0) - 2(A) + (At + B) = 5t \Rightarrow At + (B - 2A) = 6t$. Thus, $A = 6$ and $B - 2A = 0$. So $B = 12$

   (d) *General Solution*:
   $y(t) = c_1e^t + c_2te^t + 6t + 12$.

3. Give the general solution to $y'' + 4y = \cos(t)$.

Solution:

(a) *Solve Homogeneous*:
The equation $r^2 + 4 = 0$ has the roots $r = \pm 2i$.
So $y_1(t) = \cos(2t)$ and $y_2(t) = \sin(2t)$.

(b) *Particular Solution Form*:
$Y(t) = A\cos(t) + B\sin(t)$

(c) *Substitute*:
$Y'(t) = -A\sin(t) + B\cos(t)$ and $Y''(t) = -A\cos(t) - B\sin(t)$. Substituting gives $(-A\cos(t) - B\sin(t)) + 4(A\cos(t) + B\sin(t)) = \cos(t) \Rightarrow 3A\cos(t) + 3B\sin(t) = \cos(t)$.
Thus, $A = \frac{1}{3}$ and $B = 0$.

(d) *General Solution*:
$y(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{1}{3}\cos(t)$.

4. Give the general solution to $y'' - 5y' = 3e^{5t}$.

Solution:

(a) *Solve Homogeneous*:
The equation $r^2 - 5r = 0$ has the roots $r_1 = 0$, $r_2 = 5$.
So $y_1(t) = 1$ and $y_2(t) = e^{5t}$.

(b) *Particular Solution Form*:
$Y(t) = Ate^{5t}$ (because $y_2(t) = e^{5t}$).

(c) *Substitute*:
$Y'(t) = Ae^{5t} + 5Ate^{5t} = A(1 + 5t)e^{5t}$ and $Y''(t) = 5Ae^{5t} + 5A(1 + 5t)e^{5t} = A(10 + 25t)e^{5t}$.
Substituting gives
$A(10 + 25t)e^{5t} - 5A(1 + 5t)e^{5t} = 3e^{5t} \Rightarrow 5Ae^{5t} = 3e^{5t}$. Thus, $A = \frac{3}{5}$.

(d) *General Solution*:
$y(t) = c_1 + c_2e^{5t} + \frac{3}{5}te^{5t}$.

5. Give the general solution to $y'' - 3y' + 3y = 3t + e^{-2t}$.

Solution:

(a) *Solve Homogeneous*:
The equation $r^2 - 3r + 3 = 0$ has the roots $r = \frac{3 \pm \sqrt{9-12}}{2} = \frac{3}{2} \pm \frac{\sqrt{3}}{2}i$.
So $y_1(t) = e^{3t/2}\cos(\sqrt{3}t/2)$ and $y_2(t) = e^{3t/2}\sin(\sqrt{3}t/2)$.

(b) *Particular Solution Form*:
$Y(t) = At + B + Ce^{-2t}$.

(c) *Substitute*:
$Y'(t) = A - 2Ce^{-2t}$ and $Y''(t) = 4Ce^{-2t}$. Substituting gives
$4Ce^{-2t} - 3(A - 2Ce^{-2t}) + 3(At + B + Ce^{-2t}) = 3t + e^{-2t} \Rightarrow 3At + (-3A + 3B) + (4C + 6C + 3C)e^{-2t} = 3t + e^{-2t}$. Thus, $3A = 3$, $-3A + 3B = 0$ and $13C = 1$. So $A = 1$, $B = 1$, and $C = \frac{1}{13}$

(d) *General Solution*:
$y(t) = c_1 e^{3t/2}\cos(\sqrt{3}t/2) + c_2 e^{3t/2}\sin(\sqrt{3}t/2) + t + 1 + \frac{1}{13}e^{-2t}$.

6. Give the general solution to $y'' - 9y = (5t^2 - 1)e^t$.

Solution:

(a) *Solve Homogeneous*:
The equation $r^2 - 9 = 0$ has the roots $r = \pm 3$.
So $y_1(t) = e^{3t}$ and $y_2(t) = e^{-3t}$.

(b) *Particular Solution Form*:
$Y(t) = (At^2 + Bt + C)e^t$

(c) *Substitute*:
$Y'(t) = (2At + B)e^t + (At^2 + Bt + C)e^t = (At^2 + (2A + B)t + (B + C))e^t$ and
$Y''(t) = (2At + (2A + B))e^t + (At^2 + (2A + B)t + (B + C))e^t = (At^2 + (4A + B)t + (2A + 2B + C))e^t$.
Substituting gives
$(At^2 + (4A + B)t + (2A + 2B + C))e^t - 9(At^2 + Bt + C)e^t = (5t^2 - 1)e^t$
$\Rightarrow -8At^2 + (4A - 8B)t + (2A + 2B - 8C) = 5t^2 - 1$.
Thus, $-8A = 5$, $4A - 8B = 0$ and $2A + 2B - 8C = -1$. So $A = -\frac{5}{8}$, $B = \frac{1}{2}A = -\frac{5}{16}$, and $C = \frac{2A+2B+1}{8} = -\frac{5}{32} - \frac{5}{64} + \frac{1}{8} = -\frac{7}{64}$.

(d) *General Solution*:
$y(t) = c_1 e^{3t} + c_2 e^{-3t} + (-\frac{5}{8}t^2 - \frac{5}{16}t - \frac{7}{64})e^t$.