Exam 1 Review

This review sheet contains this cover page (a checklist of topics from Chapters 1 and 2). Following by all the review material posted so far this quarter in the order we used them (all combined into one file).

- Ch. 1: Differential Equation Basics
  - How do you check a solution?
  - What is a slope field? How do you recognize that a slope field matches a differential equation?
  - What are basic translation tools for applied problems?
- 2.1: Integrating Factor Method. Only for linear equations! Write in form $\frac{dy}{dt} + f(t)y = g(t)$. Then multiply by the integrating factor $e^{\int p(t)\,dt}$ and simplify. Then integrate to get your answer.
- 2.2: Separable Equations. Factor and write in the form $f(y)dy = g(x)dx$. Integrate both sides!
- 2.3: Applications:
  - Know the homework problems!
  - Be able to set up mixing problems, Newton's law of cooling, financial problems, air resistance, and population problems.
  - Know basic language and how to translate ('proportional to', 'rate').
  - For more complicated applications, the differential equation will be given. Be able to solve the equation and be able to read the question carefully to get initial conditions.
- 2.4: Existence and Uniqueness. For linear, find discontinuities of $f(t)$ and $g(t)$ (solution only guaranteed to be unique up to the discontinuities).
  For nonlinear, two things:
  - Find discontinuities of $f(t, y)$ and $\frac{\partial f}{\partial t}(t, y)$ (solution only guaranteed to exist and be unique in some interval within discontinuities).
  - Identify equilibrium solutions at the beginning (the equilibrium solutions may not be contained in the answer you get from separating and integrating).
- 2.5: Autonomous equations. Know how to find and classify equilibrium solutions (stable, unstable, semistable). Also know basic population applications.
- 2.6: Exact equations. Write in form $M(x, y) + N(x, y)\frac{dy}{dx} = 0$ and check $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. If that is true, then integrate $\int M(x, y)\,dx$ and $\int N(x, y)\,dy$ and combine to get $\psi(x, y)$. The final answer is $\psi(x, y) = c$ for some constant c.
- 2.7: Euler's method. Understand how to do Euler's method. On a test I might ask you to do 3 or 4 steps of Euler's method on a given problem. You should be able to do this.

Other skills:

- Be able to integrate! Any integration method you have seen in the homework, test preps or lecture is fair game on the test. That includes: by parts, substitution and partial fractions (and a bit of trig).
- Be able to check your work. What does it mean to have correctly found a solution to a differential equation?

Chapter 2: Summary of First Order Solving Methods

Given $\frac{dy}{dt} = f(t, y)$ with $y(t_0) = y_0$.

1. LINEAR?
   If so, rewrite in the form $\dfrac{dy}{dt} + p(t)y = g(t)$. And use the integrating factor method!

2. SEPARABLE?
   If so, factor, separate and integrate: $\dfrac{dy}{dt} = f(t, y) = h(t)g(y) \Longrightarrow \displaystyle\int \frac{1}{g(y)}\, dy = \int h(t)\, dt.$

3. EXACT?:
   Rewrite in form $M(t, y) + N(t, y)\dfrac{dy}{dt} = 0$: if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial t}$, then use the exact method.
   Integrate to get $\displaystyle\int M(t, y)\, dt + C_1(y)$ and $\displaystyle\int N(t, y)\, dy + C_2(t)$, combine to get $\psi(t, y)$ and the solution is $\psi(t, y) = C$.

4. SUBSTITUTION?:
   Let $u =$ 'some expression involving t and y', then differentiate with respect to t to get a relationship between $\dfrac{du}{dt}$ and $\dfrac{dy}{dt}$. Substitute to turn $\dfrac{dy}{dt} = f(t, y)$ into an equation involving only t and u. And HOPE! Hope that the new equation is one you can solve by one of our other methods. If I give you such a problem on the test, I will tell you the substitution to use.

Other Notes:

1. If you are asked to find an **explicit** solution, then your final answer needs to be in the form $y = y(x)$. In other words you must solve for y. If you do not (or cannot) solve for y in terms of t, then we say your answer is an **implicit** solution.

2. Remember to recognize any equilibrium solutions at the beginning. And you can also classify them as stable, unstable or semistable before you start (this also helps to check your work).

3. Remember to use your initial condition in the end!

4. If $f(t, y)$ is discontinuous or undefined at any t values, then that restricts the domain of our final answer. If $f(t, y)$ or $\dfrac{\partial f}{\partial t}(t, y)$ is discontinuous or undefined at any y, then that also restricts the domain/range of our solution. If the initial condition is at one of these discontinuities, then solutions may not exists and may not be unique.

5. You can always **check your final answer**! Here is how you check:

   (a) Take your solution and differentiate to find $\frac{dy}{dt}$. Substitute what you just found for $\frac{dy}{dt}$ in your differential equation. Also replace y by $y(t)$ in your differential equation. If both sides of the differential equation are equal, then you have a solution!

   (b) Also check your initial condition.

   (c) If your function works in the differential equation (makes both sides equal) and if your function satisfies the initial condition, then you will know with certainty that you have a solution!

Chapter 1 Review: Introduction

Essential Facts from Calculus 1 and 2:

1. $\frac{dy}{dt}$ = 'instantaneous rate of change of y with respect to t' = 'slope of the tangent line to $y(t)$ at t'.
   - $\frac{dy}{dt} = 0 \implies y(t)$ has a horizontal tangent.
   - $\frac{dy}{dt} > 0 \implies y(t)$ is increasing.
   - $\frac{dy}{dt} < 0 \implies y(t)$ is decreasing.
   - Any value of t where $\frac{dy}{dt} = 0$ is called a critical number for $y(t)$
     (we should know how to determine if a critical number corresponds to local max, local min, or neither)
2. Integration!! You must know your integration well. You will regularly use substitution and by parts throughout the quarter (and you will use partial fractions frequently in the last two weeks of the quarter).

Application Notes:

1. Translation tools:
   - 'rate of change of BLAH' $\iff \frac{d(BLAH)}{dt}$
   - '...is proportional to...' $\iff$ '...is a constant multiple of...'
   - units of $\frac{dy}{dt} \iff y$-units/t-units
2. Specific Examples:
   - "The rate of change of y is a constant c" $\iff \dfrac{dy}{dt} = c$ (Such as: Deposit an average of $c = 2000$ dollars into an account each year or 500 people immigrate into a city each year...)
   - "The rate of change of y is proportional to y" $\iff \dfrac{dy}{dt} = ky$
     (Continuously compounded interest, population growth)
   - "The rate of change of temperature an object is proportional to the difference between the object and the surrounding temperature." $\iff \dfrac{dT}{dt} = k(T - T_s)$
   - Mixing Problems $\iff \dfrac{dy}{dt} =$ (Rate IN) − (Rate OUT)
     (Note: Rate OUT will involve y in some say and watch your units!)
   - Force Diagram (Motion) Problems $\iff m\frac{dv}{dt} = F$
     (F is the sum of all the forces, v is velocity, m is mass).

Classification

1. An **ordinary differential equation** (ODE) is an equation involving derivatives relating two variables (a dependent and independent variable).

2. A **partial differential equation** (PDE) is an equation involving partial derivatives relating to more than two variables (typically, one dependent and two or three independent variables)

3. The **order** of a differential equation is the highest derivative that appears in the equation.

4. A differential equation is said to be **linear** if it only involves a linear combination of first powers of the function y and its derivatives. In other words, a linear differential equation looks like $a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots a_1(t)y' + a_0(t)y = g(t)$.
   Here are two examples of linear differential equations:
   $$t^2y' + e^ty = \sin(t) \quad \text{and} \quad 8y'' - ty' + (t^2 - \ln(t))y = e^{-t}.$$

5. We say a differential equation is **nonlinear** if it is not linear (meaning it involves some other power or function of y).
   Here are two examples on nonlinear differential questions:
   $$y' + y^2 = t^3 \quad \text{and} \quad y' + \sqrt{y} = e^y + t^2.$$

6. A linear differential equation is said to have **constant coefficients** if it only has constants (meaning no t's) in front of y and the derivatives of y.
   Here are two examples:
   $$5y' + 10y = t^2 \quad \text{and} \quad 3y'' - 2y' + 10y = \cos(t).$$

7. A linear differential equation is said to be **homogenous** if the function $g(t)$ is always 0.
   Here are two examples:
   $$5y' + 10y = 0 \quad \text{and} \quad y' + t^2y = 0.$$

Slope Fields

1. Since $\frac{dy}{dt}$ is the slope of the tangent line, we can visualize a differential equation by plugging in variables values of t and y and getting slopes (make a table). Then we can plot these slopes by drawing short line segment with the given slope. If we do this at regularly space points, then the resulting graph is called a **slope field**.

2. Facts about a slope field:

   - There will be a horizontal tangent at any (t, y) points where $\frac{dy}{dt}$ is equal to zero.
   - If there is a constant value $y = c$ which satisfies the differential equation (which means the derivative is always zero), then we say $y(t) = c$ is an **equilibrium solution**.
   - Once we know where $\frac{dy}{dt}$ is equal to zero, we can look on either side and determine if $\frac{dy}{dt}$ is positive (slopes upward) or $\frac{dy}{dt}$ is negative (slopes downward).

3. Thus, we can learn a lot of information by studying a differential equation directly. And a slope field is a useful way to depict this information.

Basic First Order Differential Equation Applications

A *differential equation* is an equation involving derivatives. When doing an applied problem, you should start by labeling all quantities and drawing a figure/diagram. Also identify the dependent and independent variables. Basic differential equation mathematical modeling facts:

'rate of change of BLAH'	means $\frac{d(BLAH)}{dt}$
'...is proportional to...'	means '...is a constant multiple of...'
match units	units of $\frac{dy}{dt}$ should be y-units/t-units

Here is a list of very basic examples which only require these translation facts. See if you can translate these into differential equations (answers on the next page).

1. "A city has 160 people. The rate of population growth is a constant 10 people per year."
2. "A city has 3000 people. The rate of population growth is proportional to the population size."
3. "A city has 2100 people. The rate of population growth is the sum of two rates: the birth/death rate and immigration rate. The birth/death rate is proportional to the population size. The rate due to immigration is a constant 100 people/year into the city."
4. "In a 1000 person town, exactly 200 people currently have a certain cold virus. Assume the rate at which the virus is spreading is proportional to the number of people that do NOT have the virus."
5. "A coffee starting at a temperature of 180°F is in a room that is a constant 70°F. The rate of cooling is proportional to the difference between the coffee temperature and the room temperature."
6. "You start with \$1000 in your retirement account. You invest an average of 5000 dollars/year. In addition, your money earns interest at a rate of 6% anually, compounded continuously."

   NOTE: "6% annually, compounded continuously" is the bank's way of saying the rate of change due to interest is proportional to the amount of money in the account (with proportionality constant 0.06).
7. "A 100 liter vat full of water contains 75 kilograms of salt . Pure water is pumped in at 5 liters/min and well mixed. Water is also coming out of the bottom of the vat at 5 liters/min."
8. "A 100 liter vat full of water contains 75 kilograms of salt . Salt water with a concentration of 3 kg/L is pumped in at 5 liters/min and well mixed. Water is also coming out of the bottom of the vat at 5 liters/min."
9. "A 100 liter vat currently contains 7 liters of water with 75 kilograms of salt. Salt water with a concentration of 3 kg/L is pumped in at 5 liters/min and well mixed. Water is also coming out of the bottom of the vat at 1 liters/min. (Note: The vat is gaining a constant $5 - 1 = 4$ liters/min of water!)"
10. "A tall circular cylindrical piece of ice (an ice core) with initial radius of 4 inches is accidentally left out and starts to melt. Assume the height remains a constant 200 inches and that the rate of change of volume is proportional to the surface area around the cylinder (i.e. ignore the top/bottom)."

1. *Translation*: If $P(t)$ = the pop. at time t, then $\frac{dP}{dt} = 10$ with $P(0) = 160$. (Constant growth)

2. *Translation*: If $P(t)$ = the pop. at time t, then $\frac{dP}{dt} = kP$ with $P(0) = 3000$.
   (This is called natural growth)

3. *Translation*: If $P(t)$ = the pop. at time t, then $\frac{dP}{dt} = kP + 100$ with $P(0) = 2100$.

4. *Translation*: If $P(t)$ = the number of people that have the virus at time t, then
   $1000 - P(t)$ = the number of people that do NOT have the virus at time t.
   And we have $\frac{dP}{dt} = k(1000 - P)$ with $P(0) = 200$.

5. *Translation*: If $T(t)$ = the temp. of the coffee at time t, then $\frac{dT}{dt} = k(T - 70)$ with $T(0) = 180$.
   (This is called Newton's Law of Cooling)

6. *Translation*: If $A(t)$ = dollars saved after t years, then $\frac{dA}{dt} = 0.06A + 5000$ with $A(0) = 1000$.

7. *Translation*: If $y(t)$ = kilograms of salt after t minutes, then
   Salt IN = (0 kg/L)(5 L/min) = 0 kg/min, Salt OUT = (y/100 kg/L)(5 L/min) = $\frac{1}{20}$ y kg/min
   Thus, $\frac{dy}{dt} = 0 - \frac{1}{20}y$ with $y(0) = 75$.

8. *Translation*: If $y(t)$ = kilograms of salt after t minutes, then
   Salt IN = (3 kg/L)(5 L/min) = 15 kg/min, Salt OUT = (y/100 kg/L)(5 L/min) = $\frac{1}{20}$ y kg/min
   Thus, $\frac{dy}{dt} = 15 - \frac{1}{20}y$ with $y(0) = 75$.

9. *Translation*: First, note that the vat starts with 7 Liters of water and the amount of water in the vat is increasing at a constant rate of 4 Liters/min (thus, the equation for the amount of water in the vat is a line!). So the amount of water in the vat at time t is given by $7 + 4t$ Liters.
   Now, if $y(t)$ = kilograms of salt after t minutes, then
   Salt IN = (3 kg/L)(5 L/min) = 15 kg/min, Salt OUT = ($y/(7+4t)$ kg/L)(1 L/min) = $\frac{y}{7+4t}$ kg/min
   Thus, $\frac{dy}{dt} = 15 - \frac{y}{7+4t}$ with $y(0) = 75$.

10. *Translation*: Let $V(t)$ = the volume of ice. Cylinder surface area (without the top/bottom) is given by $SA = 2\pi rh$. And h is a constant 200 inches, so we have $SA = 400\pi r$. The statement of the problem translates to $\frac{dV}{dt} = k400\pi r$, but it would be better to have everything in terms of V and t (instead of r which depends on V).

    Since the volume of the cylinder is $V = \pi r^2 h = 200\pi r^2$, we have that $r = \sqrt{\frac{V}{200\pi}}$. Thus, we get

    $$\frac{dV}{dt} = k400\pi\sqrt{\frac{V}{200\pi}} = 2k\sqrt{200\pi V} = C\sqrt{V} \text{ with } V(0) = 200\pi(4)^2 = 3200\pi.$$

Various Notes/Comments:

1. **Law of Natural Growth**: "The rate of change of y is proportional to y". (The model for unrestricted population growth, radioactive decay, and compound interest)

2. **Newton's Law of Cooling**: "The rate of change of temperature an object is proportional to the difference between the object and the surrounding temperature." Here k is call the cooling constant.

3. **Mixing Problems**: Here we have to find the $\frac{dy}{dt}$ = Rate IN - Rate OUT. The interesting part is the rate OUT, because it depends on the current concentration, *i.e.* always depends on y! Here is a fairly standard and full example:
   Question: A 100 gallon vat of Brine (salt-water) contains 30 kg of salt diluted in it. Let $y(t)$ be the amount of salt at time t. A brine mixture that is 25% salt (units of kg/gallon) is pumped IN at 8 gallons per minute and the mixed vat is emptied OUT at the same rate.
   Answer: IN = 8 gal/min 0.25 kg/gal = 2 kg/min, OUT = 8 gal/min $\frac{y(t)}{100}$ kg/gal = 8y/100 kg/min
   Thus, $\frac{dy}{dt} = 2 - \frac{8y}{100}$ with $y(0) = 30$.

4. **Force/Motion Problems**: By Newton's second law, $F = ma$, where F is force, m is mass, and a is acceleration. Most applied motion problems proceed as follows:
   Describe all the forces (draw a force diagram) and determine a labeling (what is positive, what is negative). Add up the forces to get the total force, F. Then let v = velocity and write the differential equation: $m\dfrac{dv}{dt} = ma = F$. Here are some common scenarios:

   (a) "A falling object near earth's surface ignoring air resistance."
   The only force is due to gravity.
   Thus, $m\dfrac{dv}{dt} = \pm mg$ (use '+' if down is considered increasing distance and '-' if down is considered decreasing distance).

   (b) "A falling object near earth's surface with air resistance."
      i. 'lower velocities': The force due to air resistance is proportional to velocity (in opposite to the direction of the object). If we label down as increasing distances and the object is only going downward (jump from a plane), then we get: $m\dfrac{dv}{dt} = mg - kv$
      ii. 'bigger objects, higher velocities': The force due to air resistance is proportional to the square of velocity (in opposite to the direction of the object). Again, labeling as before (jumping from a plane), then we get $m\dfrac{dv}{dt} = mg - kv^2$.

   (c) "Far away from the Earth's surface": Let x be the distance from the surface of the earth and R be the radius of the Earth. In terms of x, the force due to gravity is given by $F = -\frac{mgR^2}{(R+x)^2}$ (law of gravitation). Thus, we get $m\dfrac{dv}{dt} = -\dfrac{mgR^2}{(R+x)^2}$.

   (d) "Object in a liquid": Much is the same in a liquid, except there is a bouyant force as well. You still have resistance which is often modeled as a constant times velocity (same as with air resistance). The bouyant force is the weight of the liquid displaced by the object (let b be the mass of the liquid displaced by the object). In any case, you get a differential equation that looks something like $m\dfrac{dv}{dt} = mg - kv - bg$

Aside - an extra discussion about 3D motion

A full discussion of motion requires tools from Math 126 (specifically, vectors, 3D parametric equations, and 3D parametric calculus). We will not discuss 3D motion problems in this course.

However, for your own interest, let me say a few things. In Math 126, you learn to study $x = x(t)$, $y = y(t)$, and $z = z(t)$ separately in terms of time.

In general, we let $v_x(t) = \frac{dx}{dt}$, $v_y(t) = \frac{dy}{dt}$ and $v_z(t) = \frac{dx}{dt}$ be the x, y, and z velocities.

If F_x, F_y, and F_z are the x, y, and z components of the forces acting on an object, then we get the three differential equation

$$m\frac{dv_x}{dt} = F_x \ , \ m\frac{dv_y}{dt} = F_y \ , \ m\frac{dv_z}{dt} = F_z$$

Once we understand this set up, the process is the same as on the previous page.

1. "A falling object near earth's surface ignoring air resistance."
   The only force is due to gravity and it only effects the z-component.
   Thus, $m\frac{dv_x}{dt} = 0 \ , \ m\frac{dv_y}{dt} = 0 \ , \ m\frac{dv_z}{dt} = -mg.$
2. "A falling object near earth's surface with air resistance." :
   $m\frac{dv_x}{dt} = -kv_x \ , \ m\frac{dv_y}{dt} = -kv_y \ , \ m\frac{dv_z}{dt} = -mg - kv_z$

You would then solve for the functions $v_x(t)$, $v_y(t)$, and $v_z(t)$ separately.

2.1: Integrating Factors

Some Observations and Motivation:

1. The first observation is the product rule: $\frac{d}{dt}(f(t)\,y) = f(t)\frac{dy}{dt} + f'(t)y$.
   Here are a couple of quick derivative examples (we are assuming y is a function of t):
   $$\frac{d}{dt}\left(t^3\,y\right) = t^3\frac{dy}{dt} + 3t^2y \quad \text{and} \quad \frac{d}{dt}\left(e^{4t}\,y\right) = e^{4t}\frac{dy}{dt} + 4e^{4t}y.$$
   $$\textbf{Thus,} \quad f(t)\frac{dy}{dt} + f'(t)y = g(t) \quad \textbf{can rewritten as} \quad \frac{d}{dt}(f(t)y) = g(t).$$
2. The second observation (using the chain rule with $e^{F(x)}$): $\frac{d}{dt}\left(e^{F(t)}\,y\right) = e^{F(t)}\frac{dy}{dt} + F'(t)e^{F(t)}y$.

Integrating Factor Method:

If we start with $\frac{dy}{dt} + p(t)y = g(t)$ AND if we can find an antiderivative of $p(t)$, then we can use the following process:

1. First rewrite the differential equation in the form: $\frac{dy}{dt} + p(t)y = g(t)$
2. Find **any** antiderivative of $p(t)$ and write $\mu(t) = e^{\int p(t)\,dt}$
3. Multiply the entire equation by $\mu(t)$ and use the facts from above, so
   $$\frac{dy}{dt}+p(t)y = g(t) \;\; \textbf{becomes} \;\; \mu(t)\frac{dy}{dt}+p(t)\mu(t)y = g(t)\mu(t) \;\; \textbf{which becomes} \;\; \frac{d}{dt}(\mu(t)y) = g(t)\mu(t)$$
4. Integrate with respect to t and you are done! (Of course, as always, also simplify, use initial conditions and check your work)

NOTES:

1. This is a method for first order **linear** differential equations. Meaning you can only have y to the first power, and nothing else in terms of y.
2. Using the substitution idea that I introduced in the previous section, you can sometimes turn a nonlinear problem into a linear problem. Here are two examples:
   - Using $u = e^y$ on the equation $e^y\frac{dy}{dx} - xe^y = 2x$ yields the linear equation $\frac{du}{dx} - xu = 2x$.
   - Using $u = \ln(y)$ on the equation $\frac{1}{y}\frac{dy}{dx} - \frac{\ln(y)}{x} = x$ yields the linear equation $\frac{du}{dx} - \frac{u}{x} = x$.
3. A small note about the form of some answers from the textbook:
   When we are unable to integrate a function in an elementary way, you will sometimes see an answer written in the following form $\int f(x)dx = \int_{x_0}^{x} f(u)\,du + C$, where x_0 is the x-value of some initial condition.
   There is nothing scary happening here, let me give you can example to ease your mind.
   Consider $\int x^2\,dx$ and $\int_0^x u^2\,du + C$. Let me compute both:
   $$\int x^2\,dx = \frac{1}{3}x^3 + C \quad \text{and} \quad \int_0^x u^2\,du + C = \frac{1}{3}u^3\Big|_0^x + C = \frac{1}{3}x^3 + C.$$
   Notice they are the same. This gives a way to explicitly include your initial condition '+C' in writing down your final answer even if you can't integrate.

Integrating Factor Examples:

1. Find the explicit solution to $4\dfrac{dy}{dt} - 8y = 4e^{5t}$ with $y(0) = \frac{2}{3}$.

   Solution:

   (a) Rewrite: $\dfrac{dy}{dt} - 2y = e^{5t}$, so $p(t) = -2$, $g(t) = e^{5t}$.

   (b) Integrating Factor: $\int p(t)\,dt = \int -2\,dt = -2t + C$, so $\mu(t) = e^{-2t}$.

   (c) Multiply: $\frac{dy}{dt} + 2y = e^{5t}$ becomes $e^{-2t}\frac{dy}{dt} + 2e^{-2t}y = e^{3t}$ which becomes $\frac{d}{dt}\left(e^{-2t}y\right) = e^{3t}$.

   (d) Integrate: $e^{-2t}y = \int e^{3t}dt = \frac{1}{3}e^{3t} + C$, so $y = \frac{1}{3}e^{5t} + Ce^{2t}$.
   Using the initial condition gives, $\frac{2}{3} = \frac{1}{3} + C$, so $C = \frac{1}{3}$.
   For a final answer of $y = \frac{1}{3}e^{5t} + \frac{1}{3}e^{2t}$.

2. Find the explicit solution to $t\dfrac{dy}{dt} + 2y = \cos(t)$ with $y(\pi) = 1$.

   Solution:

   (a) Rewrite: $\dfrac{dy}{dt} + \dfrac{2}{t}y = \dfrac{\cos(t)}{t}$, so $p(t) = \frac{2}{t}$, $g(t) = \frac{\cos(t)}{t}$.

   (b) Integrating Factor: $\int p(t)\,dt = \int \frac{2}{t}\,dt = 2\ln|t| + C = \ln(t^2) + C$, so $\mu(t) = e^{\ln(t^2)} = t^2$.

   (c) Multiply: $\frac{dy}{dt} + \frac{2}{t}y = \frac{\cos(t)}{t}$ becomes $t^2\frac{dy}{dt} + 2ty = t\cos(t)$ which becomes $\frac{d}{dt}\left(t^2y\right) = t\cos(t)$.

   (d) Integrate: $t^2y = \int t\cos(t)dt = t\sin(t) + \cos(t) + C$ (using by parts), so $y = \frac{\sin(t)}{t} + \frac{\cos(t)}{t^2} + \frac{C}{t^2}$.
   Using the initial condition gives, $1 = 0 - \frac{1}{\pi^2} + \frac{C}{\pi^2} + C$, so $C = \pi^2 + 1$.
   For a final answer of $y = \frac{\sin(t)}{t} + \frac{\cos(t)}{t^2} + \frac{(\pi^2+1)}{t^2}$.

3. Find the explicit solution to $\cos(y)\dfrac{dy}{dt} - \dfrac{\sin(y)}{t} = t$ with $y(2) = 0$. (Hint: Start with $u = \sin(y)$)

   Solution:
   Using $u = \sin(y)$ we get $\frac{du}{dt} = \cos(y)\frac{dy}{dt}$, so the differential equation can be rewritten at $\frac{du}{dt} - \frac{1}{t}u = t$.
   Now we will solve this:

   (a) Rewrite: $p(t) = -\frac{1}{t}$, $g(t) = t$.

   (b) Integrating Factor: $\int p(t)\,dt = \int -\frac{1}{t}\,dt = -\ln(t) + C = \ln\left(\frac{1}{t}\right) + C$, so $\mu(t) = e^{\ln(1/t)} = \frac{1}{t}$.

   (c) Multiply: $\frac{du}{dt} - \frac{1}{t}u = t$ becomes $\frac{1}{t}\frac{du}{dt} - \frac{1}{t^2}u = 1$ which becomes $\frac{d}{dt}\left(\frac{1}{t}u\right) = 1$.

   (d) Integrate: $\frac{1}{t}u = \int 1\,dt = t + C$, so $u = t^2 + Ct$.
   Going back to y gives $\sin(y) = t^2 + Ct$.
   Using the initial condition gives, $\sin(0) = 2^2 + 2C$, so $C = -2$.
   For an answer of $\sin(y) = t^2 - 2t$, or $y = \sin^{-1}(t^2 - 2t)$.

2.2: Separation of Variables

In this section, we consider differential equations of the form

$$\frac{dy}{dx} = f(x)g(y).$$

These equations are called **separable** differential equation because the variables t and y can be factored into a product of separate functions $f(t)$ and $g(y)$ that only involve t and y, respectively.

1. How to solve separable differential equations:

   (a) SET UP: Get $\frac{dy}{dx}$ by itself. Factor/separate the other side into the form $f(x)g(y)$.

   (b) SEPARATE: Divide by $g(y)$ on both sides (and multiply by dx).

   (c) INTEGRATE: Evaluate the two integrals:

   $$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx.$$

   (d) CLEAN UP: Remember to put in a constant of integration on one side. Then solve for y if you can and write your answer in a clean way. (Perhaps re-defining your constant to make the solution look nicer)

   (e) INITIAL CONDITION: Put in your initial conditions to solve for any unknown constants (note that you get one constant from integration).

   (f) CHECK YOUR ANSWER: As always, differentiate your answer and check that it satisfies the differential equation (also double check initial conditions).

2. Important Notes

   - This is a method for solving **first order** differential equations. It works for **linear** and **nonlinear** differential equations. However, it is NOT always possible to do step 1 of the process above (you often can't separate variables).
   - If you can solve for y, then we call the final answer an **explicit** solution. So if you are asked to find the explicit solution, then your final answer will look like $y = y(x) =$ 'an expression only involving x.'
   - If you integrate and get an equation involving x and y, but you cannot solve for y, then we call that equation an **implicit** solution as it defines the relationship between x and y implicitly be an equation (which is a perfectly reasonable final answer, but not as easy to use).
   - To determine the **interval over which an explicit solution is defined** you should

     (a) Look at the final answer: Are there are domain restrictions on the function in your final answer?
     (You can sometimes see these restriction directly from the differential equation; for example you can find when the differential equation will give vertical tangents)

     (b) The interval will contain the x value of the initial condition!

     (c) The interval over which the explicit solution is defined will be the largest interval around the initial condition that satisfies the restrictions from the final answer.

Separable Examples:

1. Find the explicit solution to $\dfrac{dy}{dx} = \dfrac{x}{y^2}$ with $y(0) = 2$.

   Abbreviated Solutions:
   $\displaystyle\int y^2 dy = \int x dx$ becomes $\frac{1}{3}y^3 = \frac{1}{2}x^2 + C$. Thus, $y = \left(\frac{3}{2}x^2 + D\right)^{1/3}$ where $D = 3C$.

   Using the initial condition gives $D = 8$, for a final explicit solution of $y = \left(\dfrac{3}{2}x^2 + 8\right)^{1/3}$.

   Note: This function is defined for all values of x.

2. Find the explicit solution to $\dfrac{dy}{dx} = 6xy^2$ with $y(0) = \frac{1}{12}$.

   Abbreviated Solutions:
   $\displaystyle\int \frac{1}{y^2} dy = \int 6x dx$ becomes $-\frac{1}{y} = 3x^2 + C$. Thus, $y = -\frac{1}{3x^2+C}$.
   Using the initial condition gives $C = -12$, for a final explicit solution of $y = -\dfrac{1}{3x^2 - 12}$.
   Note: This function is undefined at $x = \pm 2$. Since the initial condition was between these values, the interval over which the explicit solution is defined is $-2 < x < 2$.

3. Find the explicit solution to $\dfrac{dy}{dx} = x\cos(x)\cos^2(y)$ with $y(0) = 0$.

   Abbreviated Solutions:
   $\displaystyle\int \frac{1}{\cos^2(y)} dy = \int x\cos(x) dx$ becomes $\displaystyle\int \sec^2(y) dy = x\sin(x) - \int \sin(x) dx$ (identity and by parts).
   Integrating gives $\tan(y) = x\sin(x) + \cos(x) + C$.
   Solving for y gives, $y = \tan^{-1}(x\sin(x) + \cos(x) + C)$.
   Using the initial condition gives $0 = \tan^{-1}(0 + 1 + C)$, so $C = -1$, for a final explicit solution of $y = \tan^{-1}(x\sin(x) + \cos(x) - 1)$. *Note*: This function is defined for all values of x.

4. Find an explicit solution to $\dfrac{dy}{dx} = \dfrac{y}{x^2 - 2x}$ with $y(8) = -\sqrt{3}$.

   Abbreviated Solutions:
   $\displaystyle\int \frac{1}{y} dy = \int \frac{1}{x(x-2)} dx$ becomes $\displaystyle\ln|y| = \int \frac{-1/2}{x} + \frac{1/2}{x-2} dx$ (partial fractions)
   Integrating gives $\ln|y| = -\frac{1}{2}\ln|x| + \frac{1}{2}\ln|x-2| + C$.
   Logarithm rules: $\ln|y| = \ln\left(\frac{1}{\sqrt{x}}\right) + \ln\left(\sqrt{x-2}\right) + C$ becomes $\ln|y| = \ln\left(\sqrt{\frac{x-2}{x}}\right) + C$
   Exponentiating both sides gives $y = \pm e^C\sqrt{\frac{x-2}{x}}$. Letting $D = \pm e^C$, we get a cleaner looking answer of $y = D\sqrt{\frac{x-2}{x}} = D\sqrt{1 - \frac{2}{x}}$.
   Using the initial condition gives $-\sqrt{3} = D\sqrt{1 - \frac{2}{8}} = D\sqrt{3/4} = D\sqrt{3}/2$, so $D = -2$, for a final explicit solution of $y = -2\sqrt{1 - \dfrac{2}{x}}$.
   Note: This function is defined if $1 - \frac{2}{x} \geq 0$, which gives $x \geq 2$. (This function is also defined when $x < 0$, but the given initial condition has a value of x bigger than 2). The interval over which this explicit solution is defined is $x \geq 2$.

Substitution

Sometimes when we encounter a differential equation that is not separable, we can perform a change of variable that makes it separable. The idea is to replace an expression involving y by some other variable v in order to get a differential equation involving $\frac{dv}{dx}$ that is separable. Here is a rough outline (there are more sophisticated substitution methods, but we'll save that for another class):

How to solve differential equations of the form $\dfrac{dy}{dx} = f(x, y)$ using substitution:

1. CHOOSE YOUR SUBSTITUTION: Let $v = g(x, y) =$ 'some expression from $f(x, y)$' that you wish to substitute for (see below for advise on substitution).

2. SET UP: Compute $\frac{dv}{dx} = ???$ (this will require implicit differentiation). Somewhere in this derivative you will see $\frac{dy}{dx}$, replace it with $f(x, y)$ and use your definition of v to get rid of y.

3. HOPE: Hope this set up ended with a new differential equation that only involves v and x and is separable.

4. SOLVE: Solve the separable equation.

5. REPLACE: Replace v in your final answer to get a solution involving x and y.

Notes and Examples: Here are some common choices for v with examples.

- Let $v = ax + by + c$.
  For example: $\dfrac{dy}{dx} = 2x - 5y$ is not separable.
  Let $v = 2x - 5y$.
  Differentiate with respect to x to get $\frac{dv}{dx} = 2 - 5\frac{dy}{dx}$.
  Replacing $\frac{dy}{dx}$ gives $\frac{dv}{dx} = 2 - 5(2x - 5y) = 2 - 5v$. This is separable!
  BUT, this example can also be fairly easily done using integrating factors which we will learn in section 2.1.

- Let $v = \frac{y}{x}$.
  For example: $\dfrac{dy}{dx} = \dfrac{y}{x} - e^{y/x}$ is not separable.
  Let $v = \frac{y}{x}$ and rewrite it as $y = xv$.
  Differentiate with respect to x to get $\frac{dy}{dx} = v + x\frac{dv}{dx}$.
  Replacing $\frac{dy}{dx}$ gives $\frac{y}{x} - e^{y/x} = v + x\frac{dv}{dx}$ which can be written as $v - e^{v} = v + x\frac{dv}{dx}$.
  This can be simplified to the form $\frac{dv}{dx} = -\frac{e^{v}}{x}$, which is separable!
  This particular substitution is something you might see in a later course.

- Often trying $v =$ INSIDE function is something to try. For example, if I saw $(x + y)^2$, I might try $v = x + y$.

General Comment: One could investigate this method more and build rules for good choices of v, but we will not focus on this method in this class (you will use it more in Math 309). I simply wanted to show you how to change variables and give you another option to try to make an equation separable. If you are stumped on a problem, this is something to try!

2.3: Modeling with Differential Equations

Some General Comments:
A **mathematical model** is an equation or set of equations that mimic the behavior of some phenomenon under certain assumptions/approximations. Phenomena that contain rates/change can often be modeled with differential equations.
Disclaimer: In forming a mathematical model, we make various assumptions and simplifications. I am never going to claim that these models perfectly fit physical reality. But mathematical modeling is a key component of the following scientific method:

1. We make assumptions (a hypothesis) and form a model.
2. We mathematically analyze the model. (For differential equations, these are the techniques we are learning this quarter).
3. If the model fits the phenomena well, then we have evidence that the assumptions of the model might be valid.
   If the model fits the phenomena poorly, then we learn that some of our assumptions are invalid (so we still learn something).

Concerning this course:
I am not expecting you to be an expect in forming mathematical models. For this course and on exams, I expect that:

1. You understand how to do all the homework! If I give you a problem on the exam that is very similar to a homework problem, you must be prepared to show your understanding.
2. Be able to translate a basic statement involving rates and language like in the homework. See my previous posting on applications for practice (and see homework from 1.1, 2.3, 2.5 and throughout the other assignments).
3. If I give an application on an exam that is completely new and involves language unlike homework, then I will give you the differential equation (you won't have to make up a new model). In these cases, you are expected to be able to read carefully, solve/analyze the equation, and use any given initial conditions.

Case Studies:
On the following pages, I discuss a few basic applications in more detail. Some of this information is for your own interest, but most of it should help you a deeper understanding of these models.

Mixing Problems

"The rate of change of a substance in a system equals the rate at which it enters the systems minus the rate at which it exits the system".

1. *The model*: $\dfrac{dy}{dt}$ = Rate IN - Rate OUT.

2. *Comment*: There is no physical assumption being made (yet).
   If a substance is entering a system at 5 g/min and leaving at 3 g/min, then the overall rate of change within the system is $5 - 3 = 2$ g/min. That isn't an assumption, it is a fact!

3. *Set Up:*

   (a) Label/Identify: $y = y(t) =$ 'amount of substance at time t'. Identify concentrations and rates INTO the system. Identify concentrations and rates OUT of the system. Identify the initial amount of the substance (initial condition).

   (b) Volume: If the overall volume of the system is changing, find a formula for the volume at time t.

   (c) Rates: Units of dy/dt should be y-units/t-units. Please check your units! Often you are given concentrations and rate information in which case:

   - Rate IN = (concentration coming in)(fluid-in rate). In Math 125, these were constants. In this course, we may give a function $f(t)$ for a changing concentration.
   - Rate OUT = (concentration of system)(fluid-out rate) $= \left(\dfrac{y}{V(t)}\right)$(rate out),
     where $V(t)$ = volume of fluid in the system. Since $y = y(t)$ is the current amount of substance, the fraction $\frac{y}{V(t)}$ gives the current concentration in the system at time t.
     (This does assume the system is well mixed, we always make this assumption in these problems)

4. *Behavior/Solutions*: In Math 125, we only did problems where rate in was constant and volume was constant (in which case we got a separable equation). We can now do problems where rate in and/or volume is not constant. These models often give interesting asymptotic behavior. We often ask about the concentration as $t \to \infty$.

Temperature Models

Newton's Law of Cooling: "The rate of change of temperature an object is proportional to the difference between the object and the surrounding temperature."
I have been told that this model is good for heat transfer through *convection* and it models well the rate of change in temperature between a solid and a fluid/gas. I have also been told it is more accurate when the temperature differences are small.

1. *The model*: $\frac{dT}{dt} = k(T - T_s)$, where T_s is temperature of the surroundings and $T = T(t)$ is the temperature of the object.

2. *Comments*: This model is sometimes used when the temperature of the surroundings is variable, in which case $T_s = T_s(t)$ would need to be given and the equation will not be separable. In Math 125, we only discussed cases in which T_s was constant, in which case you get a separable equation.

3. *Constants*: The constant k is called the cooling constant and is dependent on the object, surroundings, and container. For cooling coffee, a k closer to zero would mean you have a mug with better insulation.

4. *General Behavior*: If T_s is constant, there is an equilibrium solution at $T(t) = T_s$ and all starting temperatures (above or below this temperature) will give functions that tend toward T_s as $t \to \infty$.

5. *Solutions*: If T_s is constant, we can separate and solve to get $T(t) = T_s + (T_0 - T_s)e^{kt}$. You should know how I got this!. So we see that this assumption leads to an exponential function for temperature.

Stefan-Boltzmann law: "The rate of change of temperature of an object due to radiation is proportional to the difference between the fourth power of the object's temperature and the fourth power of the surrounding temperature."
I have been told that this model is good for heat transfer through *radiation*. An example is a red hot piece of iron radiating heat. In these scenarios typically the difference in temperature is very large.

1. *The model*: $\frac{dT}{dt} = k(T^4 - T_s^4)$, where T_s is temperature of the surroundings and $T = T(t)$ is the temperature of the object. To simplify the problem, in the homework you will assume that T_s is small relative to T and the equation will become $\frac{dT}{dt} = kT^4$.

2. *Constants*: The constant k is the cooling constant and it depends on the material of the object and surroundings.

3. *General Behavior*: Similar to Newton's law of cooling in terms of equilibrium and long term behavior.

4. *Solutions*: You'll do this in homework for the simplified version. It is NOT exponential, it is a rational function.

NOTE: To truly study heat transfer, you need the heat equation which you will study in Math 309.

Financial Models

Interest bearing accounts where "interest is compounded continuously"

1. *The No Interest Model*: If you put money into a jar (or under your mattress), then you don't earn any interest. If you approximate that you put in \$500 each year, then the model for the amount of money saved will be 'rate of change due to deposits' = 500. In general, 'the rate due to deposits/payments' = P = the annual amount we pay/deposit.

2. *The Amount with Interest Model*: When a banker says "interest has an annual decimal rate of r, compounded continuously", then they are saying that interest is proportional to the amount in the account. Meaning 'the rate due to interest' = ry.
   If we invest a lump sum of money $y(0) = y_0$ into an account paying an annual decimal rate of r, compounded continuously and we NEVER deposit or pay any money, then we get the model $\frac{dy}{dt} = ry$ with a solution of $y(t) = y_0 e^{rt}$ for the amount of money in the account after t years.

3. *Compound Interest AND regular deposits/payment*: Typical in saving money or paying off a debt, there is interest AND regular deposits/payments.
   Putting this all together: $\frac{dy}{dt} = ry \pm P$ dollars/year
   (Note: The $\pm$ would be positive if the deposits are adding to the account balance and negative if you are making payments that decrease the balance of the account)

4. *Comments*: If you take a finance/accounting class in investments or mortgage, you will find that there are precise formulas that can generate payments schedules to the nearest penny. But these formulas are messy and a bit tedious to use. In comparison, the model above is easy to set up and gives a formula that's easy to use (and it gives a very good approximation). Since personal depositing and payments aren't typically precise, the models above can be used to get very good estimates for interest bearing account balances.

5. *Behavior/Solutions*: Solutions are exponential. Compound interest leads to exponential growth. Put away a bit of money each month with a reasonable interest rate and it can grow to be a very large sum over 20-30 years, that's the power of compound interest! (It's exponential!)

Force/Motion Problems

By Newton's second law, $F = ma$, where F is force, m is mass, and a is acceleration.
Since $a = \frac{dv}{dt}$, we get $m\frac{dv}{dt} = F =$ 'the sum of all the forces on the object'.

1. *Labeling*: Always draw a force diagram. Decide if you want upward velocities to be positive or negative (this is your choice!), but then be consistent throughout the rest of the problem.

2. *No air resistance*: "A falling object near earth's surface ignoring air resistance."
   The only force is due to gravity. Thus, $\frac{dv}{dt} = \pm mg$, where $g = 9.8$ m/s^2 $= 32$ ft/s^2.

3. *Air resistance models*: "A falling object near earth's surface with air resistance."

   (a) The force due to air resistance is called the **drag** force. It is always in the direction *opposite* of velocity.

   (b) One model for air resistance (good for smaller objects) is: "The force due to air resistance is proportional to velocity."
   Thus, 'drag force' $= \pm kv$, where k is called the drag coefficient. This sets up nicely if we let downward motion be positive velocity (in which case, drag = -kv is negative) and upward motion be negative velocity (in which case, drag = -kv is positive). With this labeling we get $m\frac{dv}{dt} = mg - kv$ is valid for all velocities (up or down).

   (c) Another model for air resistance (good for larger objects) is: "The force due to air resistance is proportional to the square of velocity." In this case, if we let downard motion be positive velocity, then we get:
   $m\frac{dv}{dt} = mg - kv^2$ if the object is moving downward (most common use)
   $m\frac{dv}{dt} = mg + kv^2$ if the object is moving upward.

   (d) *Comment*: Terminal velocity is the equilibrium velocity in an air resistance model (it is the velocity that you approach over time as the force due to gravity balances out with the force due to drag).

4. *Object in a liquid*: Much is the same in a liquid, except there is a bouyant force. The resistance is often modeled as a constant times velocity (same as with air resistance). The bouyant force is the weight of the liquid displaced by the object (let b be the mass of the liquid displaced by the object). In any case, you get a differential equation that looks something like $m\frac{dv}{dt} = mg - kv - bg$

5. *Changing gravity,* i.e. *objects far away from the Earth's surface*: Let x be the distance from the surface of the earth and R be the radius of the Earth. In terms of x, the force due to gravity is given by $F = -\frac{mgR^2}{(R+x)^2}$ (law of gravitation). For small values of x (near the Earth's surface) notice that $-\frac{mgR^2}{(R+x)^2} \approx -mg$. Thus, we get $m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2}$. Note: x, v, and t depend on each other, we need to simplify/rewrite this equation in terms of only two variables:
   By definition, note that $v = \frac{dx}{dt}$. And the chain rule gives $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$. With this, we can rewrite the differential equation in terms of v and x to look like $mv\frac{dv}{dx} = -\frac{mgR^2}{(R+x)^2}$.
   The book shows how to find *escape* velocity which is the intial velocity needed to completely escape the Earth's pull due to gravity.

2.4: First Order Theorems about Existence and Uniqueness

In this section you learn some basic theorems about when a solution will exist and be unique. These theorems are of practical and theoretical importance. Ultimately, these theorems give you a few 'trouble' spots to check before you start solving a differential equation. By 'trouble' spots, I mean initial values and locations where you can get no solution or multiple solutions. We certainly want to know that a solution exists and that the solution we find is the only solution!
Here are some rough, practical, oversimplified guides to how to approach first order problems in order to identify and avoid 'unusual' behavior.

NONLINEAR Initial Analysis Checklist
Given **any** first order equation of the form (linear or **nonlinear**): $\dfrac{dy}{dt} = f(t, y)$ with $y(t_0) = y_0$.

1. Identify any discontinuities of $f(t, y)$ or $\dfrac{\partial f}{\partial y}(t, y)$.
   At each discontinuity t-value draw a vertical line.
   At each discontinuity y-value draw a horizontal line.
   Solutions with the initial condition $y(t_0) = y_0$ are only guaranteed to exist and be unique around (t_0, y_0) and up to when the solution reaches the nearest horizontal or vertical 'discontinuity' line.

2. Note any equilibrium solutions.
   If you divide by a quantity when you separate, then make a note that the division is only valid if the quantity is NOT zero. This is vital for nonlinear systems. Sometimes, for nonlinear systems the equilibrium solution is NOT contained in the general solution you find from separating and integrating.

3. Then solve like you normally do. Separate and integrate.

LINEAR Initial Analysis Checklist
Given a **linear** first order equation of the form: $\dfrac{dy}{dt} + p(t)y = g(t)$ with $y(t_0) = y_0$.

1. Identify any discontinuities of $p(t)$ and $g(t)$.
   At each t-value corresponding to a discontinuity draw a vertical line.
   Solutions with the initial condition $y(t_0) = y_0$ are guaranteed to be valid around (t_0, y_0) and up to when the solution reaches the nearest vertical 'discontinuity' line.

2. It's good to note any equilibrium solutions, but it is not vital like in the nonlinear case.
   Because if you use the integrating factor method then you won't be dividing by y. And the equilibrium solution will be contained in the general solution!

3. Then solve like you normally do. Integrating factor!

NOTE: In this course we don't prove the facts from this section. Since $\dfrac{dy}{dt} = f(t, y)$, we might expect that at any discontinuity of $f(t, y)$ or $\frac{\partial f}{\partial y}(t, y)$ the slope field or the solution will either be undefined or change in some sort of dramatic way (which could lead to no solution or many solutions). For more explanations of the underpinnings of the theory, read 2.4 and 2.8 (and take higher level courses on differential equations).

Some Random Examples:

1. Consider $(t-2)\dfrac{dy}{dt} + y = \dfrac{t-2}{t}$ with $y(1) = 4$.

   (a) This is a LINEAR equation in y. Divide by $(t-2)$ to get: $\dfrac{dy}{dt} + \dfrac{1}{t-2}y = \dfrac{1}{t}$.

   (b) Discontinuities? At $t = 0$ and $t = 2$. Since the initial condition is at $t = 1$, the theorems from this section only guarantee a unique solution in the interval $0 < t < 2$.

   (c) Solution: $y(t) = \frac{t-2\ln(t)-5}{t-2}$ which is indeed valid and unique in the interval $0 < t < 2$.

2. Consider $\dfrac{dy}{dt} + \dfrac{1}{\cos^2(t)}y = e^{-\tan(t)}$ with $y(\pi) = 2$.

   (a) This is a LINEAR equation in y. It is already in the correct form.

   (b) Discontinuities? $\frac{1}{\cos^2(x)}$ has a discontinuity at all values at which $\cos(t) = 0$ (these are the same discontinuities that $\tan(t)$ has). The discontinuities are $t = \ldots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$.
   The initial condition is at $t = \pi$, so the theorems from this section only guarantee a unique solution in the interval $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

   (c) Solution: $y(t) = (t+2-\pi)e^{-\tan(t)}$ which is indeed valid and unique in the interval $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

3. Consider $\dfrac{dy}{dt} = \dfrac{1}{y(t-3)^2}$ with $y(1) = -4$.

   (a) This is a NONLINEAR equation with $f(t,y) = \frac{1}{y(t-3)^2}$ and $\frac{\partial f}{\partial t}(t,y) = -\frac{1}{y^2(t-3)^2}$

   (b) Discontinuities? At $t = 3$ and $y = 0$. Since our initial condition is at $t_0 = 1$ and $y_0 = -4$, the theorems from this section only guarantee a unique solution in the interval around t_0 for which the solution stays in the rectangular region $-\infty < t < 3$ and $-\infty < y < 0$.

   (c) Equilibrium? No equilibrium values.

   (d) General Solution: $y(t) = \pm\sqrt{\dfrac{-2}{t-3} + C}$.

   The initial condition $y(1) = -4$ gives $y(t) = -\sqrt{\dfrac{-2}{t-3} + 15}$, which is only guaranteed to be unique when the function satisfies $-\infty < t < 3$ and $-\infty < y < 0$.

4. Consider $\dfrac{dy}{dt} = 2t(y-1)^{4/5}$ with $y(1) = 2$.

(a) This is a NONLINEAR equation with $f(t,y) = 2t(y-1)^{4/5}$ and $\frac{\partial f}{\partial t}(t,y) = \frac{8}{5}t(y-1)^{-1/5}$.

(b) Discontinuities? At $y = 1$. Since our initial condition is at $t_0 = 1$ and $y_0 = 2$, the theorems from this section only guarantee a unique solution in the interval around t_0 for which the solution stays in the rectangular region $-\infty < t < \infty$ and $1 < y < \infty$.

(c) Equilibrium? There is an equilibrium solution at $y = 1$.

(d) Separating and solving and you get 'general' answers of the form: $y(t) = 1 + \left(\dfrac{1}{5}t^2 + C\right)^5$.

The initial condition $y(1) = 2$ gives $y(t) = 1 + \left(\dfrac{1}{5}t^2 + \dfrac{4}{5}\right)^5$, which is only guaranteed to be unique when the function satisfies $-\infty < t < \infty$ and $1 < y < \infty$.

NOTE: If the initial condition is of the form $y(t_0) = 1$, then you are not guaranteed unique solutions because your initial condition $y_0 = 1$ is one of the points of discontinuity.
For example, consider $\dfrac{dy}{dt} = 2t(y-1)^{4/5}$ with $y(0) = 1$.
Using this initial condition in our 'general' answer gives $C = 0$ for a solution of $y(t) = 1 + \left(\frac{1}{5}t^2\right)^5 = 1 + \frac{1}{5^5}t^{10}$. This is indeed a solution (it satisfies the differential equation and this initial condition). However, there is another solution, the equilibrium solution $y(t) = 1$ which also satisfies the differential equation and the initial condition. So the solution is not unique!

5. Consider $\dfrac{dy}{dt} = y^3\sqrt{t}$ with $y(1) = \frac{1}{3}$.

(a) This is a NONLINEAR equation with $f(t,y) = 3y^3\sqrt{t}$ and $\frac{\partial f}{\partial t}(t,y) = 9y^2\sqrt{t}$.

(b) Discontinuities? We must have $t \geq 0$. Since our initial condition is at $t_0 = 1$ and $y_0 = 4$, the theorems from this section only guarantee a unique solution in the interval around t_0 for which the solution stays in the rectangular region $0 < t < \infty$ and $-\infty < y < \infty$.

(c) Equilibrium? There is an equilibrium solution at $y = 0$.

(d) Separating and solving and you get 'general' answers of the form: $y(t) = \dfrac{\pm 1}{\sqrt{C - 4t^{3/2}}}$.

The initial condition $y(1) = \frac{1}{3}$ gives $y(t) = \dfrac{1}{\sqrt{13 - 4t^{3/2}}}$, which is only guaranteed to be unique when the function satisfies $0 < t < \infty$ and $-\infty < y < \infty$.

NOTE: Don't forget the equilibrium solution when doing a problem like this. For example, consider $\dfrac{dy}{dt} = 3y^3\sqrt{t}$ with $y(1) = 0$. If you use this in our 'general' answer ($y(t) = \dfrac{\pm 1}{\sqrt{C - 4t^{3/2}}}$), you get $0 = \frac{\pm 1}{C-4}$ which has NO solution for C!!!
Our theorem guarantees that a solution exists and that it is unique, so where did our solution go? The answer is that the 'general' answer I gave did not contain all the answers. In a nonlinear problem, the equilibrium solutions may not be contained in the answer you get from separating and integrating. So the correct general answer for this differential equation is:
$y(t) = 0$ OR $y(t) = \dfrac{\pm 1}{\sqrt{-t^{3/2} + C}}$.
For the initial condition $y(1) = 0$, the unique solution is the equilibrium solution $y(t) = 0$.

2.5: Autonomous Differential Equations and Equilibrium Analysis

An **autonomous** first order ordinary differential equation is any equation of the form: $\dfrac{dy}{dt} = f(y)$.
Note: In my home dictionary, the word "autonomous" is defined as "existing or acting separately from other things or people". In the context of differential equations, autonomous means that the derivative can be expressed without any explicitly reference to time, t.

Solving Autonomous Equations:
Since there is no time t, ALL autonomous equations are separable! In other words, separating gives $\frac{1}{f(y)}dy = dt$ and integrating gives

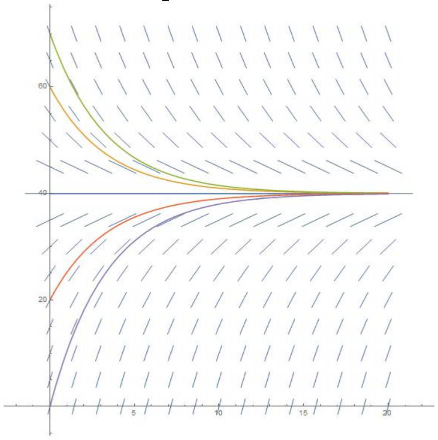
$$\int \frac{1}{f(y)} dy = t + C.$$

Of course, we would need to be able to compute this integral!

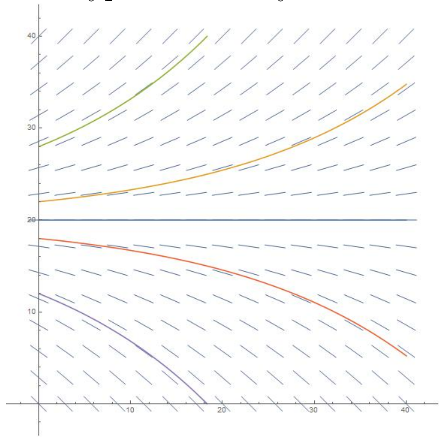
Stable, Unstable and Semi-stable Equilibrium Solutions:
Recall that an **equilibrium solution** is any constant (horizontal) function $y(t) = c$ that is a solution to the differential equation. Notice that the derivative of a constant function is always 0, so we find equilibrium solutions by solving for y in the equation $\frac{dy}{dt} = f(t, y) = 0$.
For autonomous equations, using the general existence/uniqueness theorem of the last section, if $f(y)$ and $\frac{\partial f}{\partial y}$ have no discontinuities, then we are guaranteed that no other solution can intersect an equilibrium solution (if they did we wouldn't have uniqueness).
Thus, for situations when $f(y)$ and $\frac{\partial f}{\partial y}$ have no discontinuities, we can always classify the equilibrium solutions as follows:

- **Stable**: The equilibrium solution $y(t) = c$ is **stable** if all solutions with initial conditions y_0 'near' $y = c$ approach c as $t \to \infty$.
- **Unstable**: The equilibrium solution $y(t) = c$ is **unstable** if all solutions with initial conditions y_0 'near' $y = c$ do NOT approach c as $t \to \infty$.
- **Semi-stable**: The equilibrium solution $y(t) = c$ is **semistable** if initial conditions y_0 on one side of c lead to solutions $y(t)$ that approach c as $t \to \infty$, while initial conditions y_0 on the other side of c do NOT approach c.

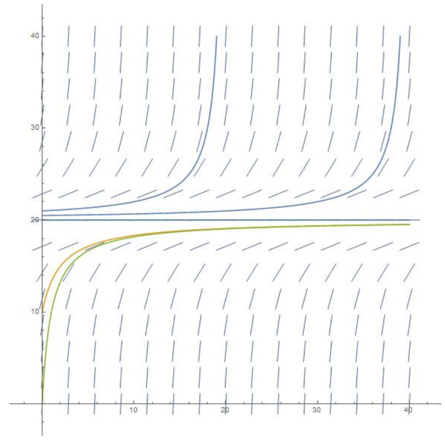
A basic example and illustration of each type of stability is shown below:



$y(t) = 40$ is STABLE
$\frac{dy}{dt} = -0.3(y - 40)$



$y(t) = 20$ is UNSTABLE
$\frac{dy}{dt} = 0.3(y - 20)$



$y(t) = 20$ is SEMISTABLE
$\frac{dy}{dt} = 0.05(y - 40)^2$

Classifying Equilibrium Solutions:

Given $\dfrac{dy}{dt} = f(y)$. (and assuming f and $\frac{\partial f}{\partial y}$ are continuous)

1. Solve $f(y) = 0$ to get the equilibrium solutions.
2. Study $f(y)$ around the equilibrium values as follows (drawing $f(y)$ might help):
   (a) Draw a vertical line (the *phase* line) and make tick marks at equilibrium values.
   (b) Between tick marks determine if $f(y)$ is positive or negative.
   (c) If $f(y)$ is **positive**, then $\frac{dy}{dt}$ will be positive so any solution in this region will be **increasing**.
   (d) If $f(y)$ is **negative**, then $\frac{dy}{dt}$ will be negative so any solution in this region will be **decreasing**.
3. Classify:
   (a) Increasing below and decreasing above $\Longrightarrow$ stable.
   (b) Decreasing below and increasing above $\Longrightarrow$ unstable.
   (c) Decreasing below and above OR increasing below and above $\Longrightarrow$ semistable.

Examples:

1. Find and classify the equilibrium points of $\frac{dy}{dt} = (1-y)(3-y)$.
   (a) $y(t) = 1$ and $y(t) = 3$ are the equilibrium solutions.
   (b) For $y > 3$: $\frac{dy}{dt}$ is positive, so $y(t)$ is increasing.
   For $1 < y < 3$: $\frac{dy}{dt}$ is negative, so $y(t)$ is decreasing.
   For $y < 1$: $\frac{dy}{dt}$ is positive, so $y(t)$ is increasing.
   (c) Thus, $y(t) = 1$ is stable. And $y(t) = 3$ is unstable.
2. Find and classify the equilibrium points of $\frac{dy}{dt} = -(y-10)^2(y-4)$.
   (a) $y(t) = 4$ and $y(t) = 10$ are the equilibrium solutions.
   (b) For $y > 10$: $\frac{dy}{dt}$ is negative, so $y(t)$ is decreasing.
   For $4 < y < 10$: $\frac{dy}{dt}$ is negative , so $y(t)$ is decreasing.
   For $y < 4$: $\frac{dy}{dt}$ is positive, so $y(t)$ is increasing.
   (c) Thus, $y(t) = 4$ is stable. And $y(t) = 10$ is semistable.
3. Find and classify the equilibrium points of $\frac{dy}{dt} = (y^3 - 8)(e^y - 1)$.
   (a) $y(t) = 0$, $y(t) = 2$ are the equilibrium solutions.
   (b) For $y > 2$: $\frac{dy}{dt}$ is positive, so $y(t)$ is increasing.
   For $0 < y < 2$: $\frac{dy}{dt}$ is negative, so $y(t)$ is decreasing.
   For $y < 0$: $\frac{dy}{dt}$ is positive, so $y(t)$ is increasing.
   (c) Thus, $y(t) = 0$ is stable. And $y(t) = 2$ is unstable.

Population Dynamics:
Typically, the rate of change of a population is only dependent on the current population size in some way. Thus, the differential equation for a population is typically time-independent (so it is autonomous). The study of populations is a big application of differential equations that we have been waiting to discuss until now.

1. Let $y = y(t)$ = the size of the population at time t.
   This could be any organism (people, people that have disease, people that know a rumor, bugs, bacteria, fish, other animals, plants, ...).

2. Thus, $\dfrac{dy}{dt}$ = rate of change of population with respect to time.
   For example, if y is in bacteria and t is in minutes, then the units of $\frac{dy}{dt}$ will be bacteria/minute. So $\frac{dy}{dt}$ can, roughly, be thought of as the number of bacteria that will be added to the population over the next minute (I say 'roughly' because it actually is the instantaneous rate which is not exactly the same as the average rate over the next minute, but they would be very, very close).

We considered three models in lecture:

- Unrestricted (Natural) Growth: $\dfrac{dy}{dt} = ry$
  - Plenty of food and space. Assumes: "The rate of change of the population proportional to the population size."
  - r = relative growth rate (decimal version of the percentage growth per time). So ry is the approximate number of people added to the population each year.
  - The solution is exponential.
  - There is only one equilibrium at $y(t) = 0$ and, for positive r, it is unstable.
- Logistic Growth: $\dfrac{dy}{dt} = r\left(1 - \dfrac{y}{K}\right)y$, with $0 < K$ and $r > 0$.
  - Initially like unrestricted growth, but there is a carrying capacity K that the population size can't exceed.
  - For positive r, there are two equilibrium at $y(t) = 0$ which is unstable and $y(t) = K$ which is stable.
- Logistic Growth with a Threshold: $\dfrac{dy}{dt} = -r\left(1 - \dfrac{y}{K}\right)\left(1 - \dfrac{y}{T}\right)y$ with $0 < T < K$, $r > 0$
  - If the population size is below a certain threshold size, T, then the population will decrease to zero. If the population is above this threshold, then the population behaves like logistic growth.
  - For positive r, there are three equilibrium at $y(t) = 0$ which is stable, $y(t) = T$ which is unstable, and $y(t) = K$ which is stable.

2.6: Exact Equations

In this class, we'll define an **exact** equation as a first order differential equation of the form:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \quad \text{where} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

In such a case, a solution exists and can be found by reversing implicit differential (existence guaranteed assuming M, N, M_y, and N_x are continuous).

Here are some motivational examples. The three examples below all start with an equation that implicitly defines a function and asks you to find $\frac{dy}{dx}$ (you should remember doing such problems in Math 124, 125, and 126). The answer for $\frac{dy}{dx}$ is given in the first line, then some observation are made which should help motivate the exact equation method.

1. $x^2 + y^2 = 4$. Find $\frac{dy}{dx}$.

   Solution: Differentiating with respect to x gives: $2x + 2y\frac{dy}{dx} = 0$. Thus, $\frac{dy}{dx} = -x/y$.

   - $2x + 2y\frac{dy}{dx} = 0$ is an exact equation with $M(x,y) = 2x$ and $N(x,y) = 2y$. Observe $\frac{\partial M}{\partial y} = 0$ and $\frac{\partial N}{\partial x} = 0$ are the same.
   - $\int M(x,y)\,dx = \int 2x\,dx = x^2 + C_1(y)$ and $\int N(x,y)\,dy = \int 2y\,dy = y^2 + C_2(x)$
     Notice that the original equation is of the form $x^2 + y^2 = C$ (with $C = 4$).

2. $4x^3 + xy^4 = 7$. Find $\frac{dy}{dx}$.

   Solution: Differentiating with respect to x gives: $12x^2 + y^4 + 4xy^3\frac{dy}{dx} = 0$. Thus, $\frac{dy}{dx} = \frac{-12x^2 - y^4}{4xy^3}$.

   - $12x^2 + y^4 + 4xy^3\frac{dy}{dx} = 0$ is an exact equation with $M(x,y) = 12x^2 + y^4$ and $N(x,y) = 4xy^3$. Observe that $\frac{\partial M}{\partial y} = 4y^3$ and $\frac{\partial N}{\partial x} = 4y^3$ are the same.
   - $\int 12x^2 + y^4\,dx = 4x^3 + xy^4 + C_1(y)$ and $\int 4xy^3\,dy = xy^4 + C_2(x)$
     Notice that the original equation is of the form $4x^3 + xy^4 = C$ (with $C = 7$).

3. $3x^2e^y + 2y = 10$. Find $\frac{dy}{dx}$.

   Solution: Differentiating with respect to x gives: $6xe^y + 3x^2e^y\frac{dy}{dx} + 2\frac{dy}{dx} = 0$. Thus, $\frac{dy}{dx} = \frac{-6xe^y}{3x^2e^y+2}$.

   - $6xe^y + (3x^2e^y + 2)\frac{dy}{dx} = 0$ is an exact equation with $M(x,y) = 6xe^y$ and $N(x,y) = 3x^2e^y + 2$. Observe that $\frac{\partial M}{\partial y} = 6xe^y$ and $\frac{\partial N}{\partial x} = 6xe^y$ are the same.
   - $\int 6xe^y\,dx = 3x^2e^y + C_1(y)$ and $\int 3x^2e^y + 2\,dy = 3x^2e^y + 2y + C_2(x)$
     Notice that the original equation is of the form $3x^2e^y + 2y = C$ (with $C = 7$).

Summary of Observations and Notes:

- If $\psi(x,y) = C$ is the original, then $\psi(x,y)$ is a 'combined' expression that can be written in BOTH the forms $\int M(x,y)\,dx + C_1(y)$ and $\int N(x,y)\,dy + C_2(x)$.
- The proof of the fact that implicit differential always leads to the form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ with $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is not part of this course (for more information take Math 126 and Math 324 and read about Clairaut's Theorem).

Exact Equations Method

1. FORM: Rewrite the equation in the form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$.
2. CHECK CONDITION: Find $\dfrac{\partial M}{\partial y}$ and $\dfrac{\partial N}{\partial x}$.
   If they are different, give up! (This method won't work to solve the equation).
   If they are the same, keep going.
3. INTEGRATE: Evaluate $\displaystyle\int M(x,y)\,dx + C_1(y)$ and $\displaystyle\int N(x,y)\,dy + C_2(x)$.
4. FINAL ANSWER: The 'combined' function that looks like both expressions of previous part is $\psi(x,y)$ (write 'overlapping terms' only once). The final answer is $\psi(x,y) = c$ for some constant c. Use initial conditions to find c.
5. CHECK: Use implicit differentiation to check that your answer is correct!

ASIDE (Another perspective):
I've noticed that students often get confused by the presentation in textbooks for how to solve for $\psi(x,y)$, so I have stated the method above in terms of integrating and combining (like you saw in the examples on the previous page). Here is another (more common, textbook) way that you will see the INTEGRATE step described in textbooks.

To illustrate the textbook method, let's use the third example from the previous page which was $6xe^y + (3x^2e^y + 2)\dfrac{dy}{dx} = 0$.

1. Integrate and write $\psi(x,y) = M(x,y)\,dx + h(y)$ where $h(y)$ is some function in terms of y. The integral will give all terms of $\psi(x,y)$ that involve the variable x.
   In the example this would look like $\psi(x,y) = \displaystyle\int 6xe^y\,dx = 3x^2e^y + h(y)$
   ($h(y)$ is the constant of integration, which might involve y since we are treating y like a constant).
2. Differentiate $\psi(x,y)$ with respect to y and equate the result to $N(x,y)$. Simplifying will give an equation of the form $h'(y) = ???$.
   Using the example, $\dfrac{\partial}{\partial y}\psi(x,y) = \dfrac{\partial}{\partial y}(3x^2e^y + h(y)) = 3x^2e^y + h'(y)$.
   We want this to equal $N(x,y)$, so we want $3x^2e^y + h'(y) = 3x^2e^y + 2$.
   In other words, we need $h'(y) = 2$.
3. Integrate with respect to y to get $h(y)$. And you are done (now you can write $\psi(x,y)$.
   Using the example, $h'(y) = 2$ means that $h(y) = 2y + C$. So the original function $\psi(x,y)$ must have the form $\psi(x,y) = 3x^2e^y + 2y$ and the final answer will be $3x^2e^y + 2y = C$ for some constant C. Notice this matches the answer on the previous page.

This is an efficient, formulaic way to cancel the 'overlap' (that is what you are doing in the middle step here), so that you don't have to integrate the same thing twice. If you take Math 324 (and some other upper level math classes), then you will use this technique again.
Use any method here that makes sense to you. Just make sure to check your work by implicitly differentiating your final answer!

Examples: (Leave your answers in implicit form)

1. Solve $\dfrac{dy}{dx} = -\dfrac{x}{y}$ with $y(0) = 4$.

   (a) FORM: $x + y\frac{dy}{dx} = 0$, so $M(x,y) = x$ and $N(x,y) = y$.

   (b) CHECK CONDITION: $\frac{\partial M}{\partial y} = 0$ and $\frac{\partial N}{\partial x} = 0$ are the same!

   (c) INTEGRATE: $\displaystyle\int x\,dx = \frac{1}{2}x^2 + C_1(y)$ and $\displaystyle\int y\,dy = \frac{1}{2}y^2 + C_2(x)$.

   (d) FINAL ANSWER: 'Combining' gives $\psi(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$.
   The general implicit answer is $\frac{1}{2}x^2 + \frac{1}{2}y^2 = C$.
   Using the initial condition of $y(0) = 4$ gives $0 + \frac{16}{2} = C$, so $C = 8$.
   Thus, $\frac{1}{2}x^2 + \frac{1}{2}y^2 = 8$ which is the same as $x^2 + y^2 = 16$.

2. Solve $(3y^2 + x^3)\dfrac{dy}{dx} = -2x - 3x^2y$ with $y(5) = 0$.

   (a) FORM: $(2x + 3x^2y) + (3y^2 + x^3)\frac{dy}{dx} = 0$, so $M(x,y) = 2x + 3x^2y$ and $N(x,y) = 3y^2 + x^3$.

   (b) CHECK CONDITION: $\frac{\partial M}{\partial y} = 3x^2$ and $\frac{\partial N}{\partial x} = 3x^2$ are the same!

   (c) INTEGRATE: $\displaystyle\int 2x + 3x^2y\,dx = x^2 + x^3y + C_1(y)$ and $\displaystyle\int 3y^2 + x^3\,dy = y^3 + x^3y + C_2(x)$.

   (d) FINAL ANSWER: 'Combining' gives $\psi(x,y) = x^2 + x^3y + y^3$.
   The general implicit answer is $x^2 + x^3y + y^3 = C$.
   Using the initial condition of $y(5) = 0$ gives $5^2 + 0 + 0 = C$, so $C = 25$.
   Thus, $x^2 + x^3y + y^3 = 25$.

3. Solve $\dfrac{dy}{dx} = \dfrac{y^2\sin(x) + 3x^2}{2y\cos(x)}$ with $y(0) = 3$.

   (a) FORM: $(-y^2\sin(x) - 3x^2) + (2y\cos(x))\frac{dy}{dx} = 0$, so $M(x,y) = -y^2\sin(x) - 3x^2$ and $N(x,y) = 2y\cos(x)$.

   (b) CHECK CONDITION: $\frac{\partial M}{\partial y} = -2y\sin(x)$ and $\frac{\partial N}{\partial x} = -2y\sin(x)$ are the same!

   (c) INTEGRATE: $\displaystyle\int -y^2\sin(x) - 3x^2\,dx = y^2\cos(x) - x^3 + C_1(y)$
   and $\displaystyle\int 2y\cos(x)\,dy = y^2\cos(x) + C_2(x)$.

   (d) FINAL ANSWER: 'Combining' gives $\psi(x,y) = y^2\cos(x) - x^3$.
   The general implicit answer is $y^2\cos(x) - x^3 = C$.
   Using the initial condition of $y(0) = 3$ gives $9 - 0 = C$, so $C = 9$.
   Thus, $y^2\cos(x) - x^3 = 9$.

2.7: Euler's Method

Given $\frac{dy}{dt} = f(t, y)$ with $y(t_0) = y_0$. The methods for finding explicit (or implicit) solutions are limited. We can solve only a small collection of special types of differential equations. In many applied problems numerical methods are essential. One of the most fundamental approximation methods is Euler's method which we describe here.

IDEA: Note that $\frac{dy}{dt}$ is the slope of the tangent line to $y(t)$. If we are given a point (t_0, y_0), we can directly evaluate $\frac{dy}{dt} = f(t_0, y_0)$ to get the slope of the tangent line at that point. In fact, we can get the equation for the tangent line $y = y_0 + f(t_0, y_0)(t - t_0)$ at that point. If we change t slightly (a small step of h, so that $t_1 = t_0 + h$) and compute the new y value from the tangent line, then we will get something very close to the actual value of the solution at that step. In terms of the formula, we get a new points $t_1 = t_0 + h$ and $y_1 = y_0 + f(t_0, y_0)h$. The idea is to repeat this process over and over again in order to find the path of the solution.

EULER'S METHOD: More formally, given $\frac{dy}{dt} = f(t, y)$ with $y(t_0) = y_0$ we approximate the path of the solution by:

1. STEP SIZE: First, we choose the step size, h, which is the size of the increments along the t-axis that we will use in approximation. Smaller increments tend to give more accurate answers, but then there are more steps to compute. We often use some value areound $h = 0.1$ in our examples in this class (but in applications $h = 0.001$ is probably a better choice).
2. COMPUTE SLOPE: Compute the slope $\frac{dy}{dt} = f(t_0, y_0)$.
3. GET NEXT POINT: The next point is $t_1 = t_0 + h$ and $y_1 = y_0 + f(t_0, y_0)h$.
4. REPEAT: Repeat the last two steps with (t_1, y_1). Then repeat again with (t_2, y_2) and repeat again and again, until you get to the desired value of t.

It might help to make a table:

t	t_0	$t_1 = t_0 + h$	$t_2 = t_1 + h$	$t_3 = t_2 + h$	$\cdots$
y	y_0	$y_1 = y_0 + f(t_0, y_0)h$	$y_2 = y_1 + f(t_1, y_1)h$	$y_3 = y_2 + f(t_2, y_2)h$	$\cdots$

Here are two quick examples:

1. Let $\frac{dy}{dt} = 2t + y$, $y(1) = 5$. Using Euler's method with $h = 0.2$ approximate the value of $y(2)$.

t	1	1.2	1.4	1.6	1.8	2.0
y	5	6.4	8.16	10.352	13.0624	16.39488

   Here are the calculations for how I filled in the table above:

   (a) $f(1,5) = 2(1) + (5) = 7$, so $y(1.2) \approx 5 + 7(0.2) = 6.4$.
   (b) $f(1.2, 6.4) = 2(1.2) + (6.4) = 8.8$, so $y(1.4) \approx 6.4 + 8.8(0.2) = 8.16$.
   (c) $f(1.4, 8.16) = 2(1.4) + (8.16) = 10.96$, so $y(1.6) \approx 8.16 + 10.96(0.2) = 10.352$.
   (d) $f(1.6, 10.352) = 2(1.4) + (10.352) = 13.552$, so $y(1.8) \approx 10.352 + 13.552(0.2) = 13.0624$.
   (e) $f(1.8, 13.0624) = 2(1.4) + (13.0624) = 16.6624$, so $y(2) \approx 13.0624 + 16.6624(0.2) = 16.39488$.

   Aside: In this case, an explicit answer can be found $y(t) = 9e^{t-1} - 2(t+1)$. And note that the actual value is $y(2) = 9e - 6 \approx 18.4645$.
   So the answer we got is within 2 (which is a pretty big error).
   If you use $h = 0.1$, then it takes 10 steps to get to $y(2)$ and you get an approximation of $y(2) \approx 17.3437$.
   If you use $h = 0.01$, then it takes 100 steps to get to $y(2)$ and you get an approximation of $y(2) \approx 18.3433$.

2. Let $\frac{dy}{dt} = \frac{2}{ty} + \ln(y)$, $y(1) = 2$. Using Euler's method with $h = 0.5$ approximate the value of $y(3)$.

t	1	1.5	2	2.5	3
y	2	2.846574	3.603831	4.383571	5.213753

   Here are the calculations for how I filled in the table above:

   (a) $f(1,2) = \frac{2}{(1)(2)} + \ln(2) \approx 1.6931$, so
   $y(1.5) \approx 2 + 1.6931(0.5) \approx 2.846574$.
   (b) $f(1.5, 2.846574) = \frac{2}{(1.5)(2.846574)} + \ln(2.846574) \approx 1.514515$, so
   $y(2) \approx 2.846574 + 1.514515(0.5) \approx 3.603831$.
   (c) $f(2, 3.603831) = \frac{2}{(2)(3.603831)} + \ln(3.603831) \approx 1.55948$, so
   $y(2.5) \approx 3.603831 + 1.55948(0.5) \approx 4.383571$.
   (d) $f(2.5, 4.383571) = \frac{2}{(2.5)(4.383571)} + \ln(4.383571) \approx 1.660363$, so
   $y(3) \approx 4.383571 + 1.660363(0.5) \approx 5.213753$.

   Aside: There is no nice solution in terms of elementary functions. Using the basic numerical solver on Mathematica gives an approximation of $y(3) \approx 5.19232$ which is very close to our rough estimate.

Integration

The following table of integrals can be used without any further justification.

$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$	
$\int \frac{1}{x} \, dx = \ln\|x\| + C$	$\int \frac{1}{ax+b} \, dx = \frac{1}{a}\ln\|ax+b\| + C$
$\int e^x \, dx = e^x + C$	$\int e^{ax} \, dx = \frac{1}{a}e^{ax} + C$
$\int \cos(x) \, dx = \sin(x) + C$	$\int \sin(x) \, dx = -\cos(x) + C$
$\int \cos(ax) \, dx = \frac{1}{a}\sin(ax) + C$	$\int \sin(ax) \, dx = -\frac{1}{a}\cos(ax) + C$
$\int \sec^2(x) \, dx = \tan(x) + C$	$\int \csc^2(x) \, dx = -\cot(x) + C$
$\int \sec(x)\tan(x) \, dx = \sec(x) + C$	$\int \csc(x)\cot(x) \, dx = -\csc(x) + C$
$\int \frac{1}{x^2+a^2} \, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$	$\int \frac{1}{x^2-a^2} \, dx = \frac{1}{2a}\ln\left\|\frac{x-a}{x+a}\right\| + C$
$\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$	$\int \frac{1}{\sqrt{x^2 \pm a^2}} \, dx = \ln\left\|x \pm \sqrt{x^2+a^2}\right\| + C$
$\int \tan(x) \, dx = \ln\|\sec(x)\| + C$	$\int \cot(x) \, dx = \ln\|\sin(x)\| + C$
$\int \sec(x) \, dx = \ln\|\sec(x)+\tan(x)\| + C$	$\int \csc(x) \, dx = \ln\|\csc(x)+\cot(x)\| + C$
$\int \sinh(x) \, dx = \cosh(x) + C$	$\int \cosh(x) \, dx = \sinh(x) + C$

Note: $\sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$ and $\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$ are functions that often appear in engineering (we'll see them a bit this quarter).

For integrals that aren't in this table, some method of integration needs to be used. Most often you will need:

- Substitution (you will use this several times in almost every homework assignment)
- By Parts (you will also use this several times in most homework assignments)
- Partial Fractions (you will use this on almost every problem in the last two weeks of the quarter).

Trig integration and trig substitution will appear less often in this course (you still may see them time to time, but their appearance will be rare). You still need to remember your trig identities and you need to recognize when these methods would be appropriate.

Basic Integration Examples for Review

The following pages contain a few standard/routine examples of substitution, by parts, and partial fractions.

Basic Substitution Examples

1. $\displaystyle\int x\cos(x^2)\,dx.$

   Solution: Let $u = x^2$, so $du = 2xdx$ (and $\frac{1}{2x}du = dx$).
   The integral becomes $\displaystyle\int \frac{1}{2}\cos(u)\,du = \frac{1}{2}\sin(u) + C = \frac{1}{2}\sin(x^2) + C.$

2. $\displaystyle\int \cos(x)e^{\sin(x)}\,dx.$

   Solution: Let $u = \sin(x)$, so $du = \cos(x)dx$ (and $\frac{1}{\cos(x)}du = dx$).
   The integral becomes $\displaystyle\int e^u\,du = e^u + C = e^{\sin(x)} + C.$

3. $\displaystyle\int x^2\sqrt{x^3+2}\,dx.$

   Solution: Let $u = x^3 + 2$, so $du = 3x^2dx$ (and $\frac{1}{3x^2}du = dx$).
   The integral becomes $\displaystyle\int \frac{1}{3}\sqrt{u}\,du = \int \frac{1}{3}u^{1/2}\,du = \frac{2}{9}u^{3/2} + C = \frac{2}{9}(x^3+2)^{3/2} + C.$

4. $\displaystyle\int \frac{(\ln(x))^3}{x}\,dx.$

   Solution: Let $u = \ln(x)$, so $du = \frac{1}{x}dx$ (and $xdu = dx$).
   The integral becomes $\displaystyle\int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}(\ln(x))^4 + C.$

5. $\displaystyle\int \frac{x}{x^2+1}\,dx.$

   Solution: Let $u = x^2 + 1$, so $du = 2xdx$ (and $\frac{1}{2x}du = dx$).
   The integral becomes $\displaystyle\int \frac{1}{2}\frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C.$

Integration by Parts

1. $\int x\cos(2x)\,dx$

   Solution: Let $u = x$ and $dv = \cos(2x)dx$. Then $du = dx$ and $v = \frac{1}{2}\sin(2x)$.
   The by parts formula gives $\dfrac{1}{2}x\sin(2x) - \displaystyle\int \frac{1}{2}\sin(2x)\,dx = \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C$.

2. $\int x^2\ln(x)\,dx$

   Solution: Let $u = \ln(x)$ and $dv = x^2dx$. Then $du = \frac{1}{x}dx$ and $v = \frac{1}{3}x^3$.
   The by parts formula gives $\dfrac{1}{3}x^3\ln(x) - \displaystyle\int \frac{1}{3}x^2\,dx = \frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3 + C$.

3. $\int x^2e^{-x}\,dx$

   Solution: Let $u = x^2$ and $dv = e^{-x}dx$. Then $du = 2xdx$ and $v = -e^{-x}$.
   The by parts formula gives $-x^2e^{-x} - \displaystyle\int -2xe^{-x}\,dx = -x^2e^{-x} + \int 2xe^{-x}\,dx$.
   Now we do by parts again with $u = 2x$ and $dv = e^{-x}dx$. Then $du = 2dx$ and $v = -e^{-x}$.
   The by parts formula gives $-x^2e^{-x} - 2xe^{-x} - \displaystyle\int -2e^{-x}\,dx = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$.

4. $\int e^x\sin(x)\,dx$

   Solution: Let $u = e^x$ and $dv = \sin(x)dx$. Then $du = e^xdx$ and $v = -\cos(x)$.
   The by parts formula gives $-e^x\cos(x) - \displaystyle\int -e^x\cos(x)\,dx = -e^x\cos(x) + \int e^x\cos(x)\,dx$.
   Now we do by parts again with $u = e^x$ and $dv = \cos(x)dx$. Then $du = e^xdx$ and $v = \sin(x)$.
   The by parts formula gives $-e^x\cos(x) + e^x\sin(x) - \displaystyle\int e^x\sin(x)\,dx$.
   Thus, we have shown $\displaystyle\int e^x\sin(x)\,dx = -e^x\cos(x) + e^x\sin(x) - \int e^x\sin(x)\,dx$, from which we can conclude that $2\displaystyle\int e^x\sin(x)\,dx = -e^x\cos(x) + e^x\sin(x) + C_0$.
   Therefore, $\displaystyle\int e^x\sin(x)\,dx = -\frac{1}{2}e^x\cos(x) + \frac{1}{2}e^x\sin(x) + C$.

Partial Fractions

1. $\displaystyle\int \frac{x-2}{(x+1)(x-4)}\,dx$

   Solution: Distinct linear terms decompose into the form $\frac{x-2}{(x+1)(x-4)} = \frac{A}{x+1} + \frac{B}{x-4}$, which can be expanded to get $x-2 = A(x-4) + B(x+1) = (A+B)x + (-4A+B)$. Thus, $A+B=1$ and $-4A+B=-2$ which we can solve to get $A=\frac{3}{5}$ and $B=\frac{2}{5}$.
   (You can use the "cover-up" method to do this faster, ask me about this if you haven't seen it).
   Thus, we get $\displaystyle\int \frac{x-2}{(x+1)(x-4)}\,dx = \int \frac{3/5}{x+1} + \frac{2/5}{x-4}\,dx = \frac{3}{5}\ln|x+1| + \frac{2}{5}\ln|x-4| + C.$

2. $\displaystyle\int \frac{3}{(x+1)^2(x-2)}\,dx$

   Solution: We have a distinct and a repeated linear term which decompose into the form
   $\frac{3}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$, which can be expanded to get
   $3 = A(x+1)(x-2) + B(x-2) + C(x+1)^2 = (A+C)x^2 + (-A+B+2C)x + (-2A-2B+C)$.
   Thus, $A+C=0$, $-A+B+2C=0$ and $-2A-2B+C=3$ which we can solve to get $A=-\frac{1}{3}$, $B=-1$, $C=\frac{1}{3}$. (Again, there are many short-cuts you can use here, ask me if you don't know them).
   $\displaystyle\int \frac{3}{(x+1)^2(x-2)}\,dx = \int \frac{-1/3}{x+1} + \frac{-1}{(x+1)^2} + \frac{1/3}{x-2}\,dx = -\frac{1}{3}\ln|x+1| + \frac{1}{x+1} + \frac{1}{3}\ln|x-2| + C.$

3. $\displaystyle\int \frac{2x+1}{x(x^2+1)}\,dx$

   Solution: We have a distinct linear and an irreducible quadratic term which decompose into the form $\frac{2x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$, which can be expanded to get $2x+1 = A(x^2+1) + (Bx+C)x = (A+B)x^2 + (C)x + (A)$. Thus, $A+B=0$, $C=2$ and $A=1$ which we can solve to get $A=1$, $B=-1$, $C=2$.
   $\displaystyle\int \frac{1}{x} + \frac{-x+2}{x^2+1}\,dx = \ln|x| - \int \frac{x}{x^2+1}\,dx + \int \frac{2}{x^2+1}\,dx = \ln|x| - \frac{1}{2}\ln|x^2+1| + 2\tan^{-1}(x) + C.$

Partial Derivatives Quick Overview

In Math 307, we sometimes see functiosn of the form $f(x, y)$. This is called a mulivariable function. It gives a third value, let's say z, for each valid value pair of values (x, y) (that is $z = f(x, y)$). In Math 126, you will spend several weeks introducing and studying such functions. In this course, we will have a few occasions where we need to find a rate of change with respect to one of the variables. We will define:

$$\frac{\partial f}{\partial x}(x,y) = f_x(x,y) = \text{'the partial derivative of } f \text{ with respect to } x\text{'}.$$
$$\frac{\partial f}{\partial y}(x,y) = f_y(x,y) = \text{'the partial derivative of } f \text{ with respect to } y\text{'}.$$

For this course, you only need to know how to compute simple partial derivatives of functions of the form $f(x, y)$.

Here is how you compute $\frac{\partial f}{\partial x}$:
Treat everything in $f(x, y)$ as a CONSTANT except x (*i.e.* treat y like a constant). Then take the derivative with respect to x.

Here is how you compute $\frac{\partial f}{\partial y}$:
Treat everything in $f(x, y)$ as a CONSTANT except y (*i.e.* treat x like a constant). Then take the derivative with respect to y.

A few basic examples:

1. If $f(x, y) = x^3 + 2y^5 + 4$, then the partial derivatives are
   $\frac{\partial f}{\partial x} = 3x^2$ Note: y is a constant so the deriv. of $2y^5$ is zero.
   $\frac{\partial f}{\partial y} = 10y^5$ Note: x is a constant so the deriv. of x^3 is zero.

2. If $f(x, y) = x^4y^3 + 8x^2y + y^4 + 5x$, then the partial derivatives are
   $\frac{\partial f}{\partial x} = 4x^3y^3 + 16xy + 5$ Note: 8 and y are coefficients of x^2, where y^4 is just a constant.
   $\frac{\partial f}{\partial y} = 3x^4y^2 + 8x^2 + 4y^3$ Note: $8x^2$ is the coefficient of y and the deriv. of y is 1.

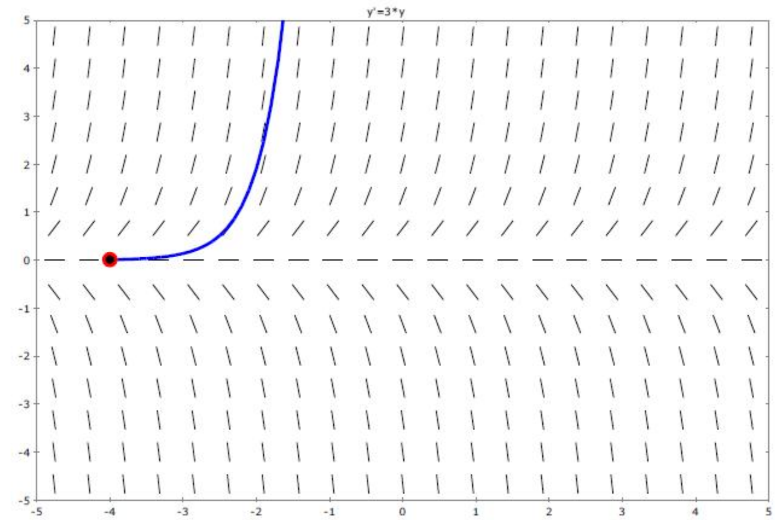
3. If $f(x, y) = \frac{x^2}{y^3} = \frac{1}{y^3}x^2 = y^{-3}x^2$, then the partial derivatives are
   $\frac{\partial f}{\partial x} = \frac{2x}{y^3}$ Note: No need for quotient rule, only an x in the numerator.
   $\frac{\partial f}{\partial y} = -3y^{-4}x^2$ Note: Again, no need for quotient rule, only a y in the denominator.

4. If $f(x, y) = (x^2 + y^3)^{10} + \ln(x)$, then the partial derivatives are
   $\frac{\partial f}{\partial x} = 20x(x^2 + y^3)^9 + \frac{1}{x}$ Note: We used the chain rule on the first term.
   $\frac{\partial f}{\partial y} = 30y^2(x^2 + y^3)^9$ Note: Chain rule again, and second term has no y.

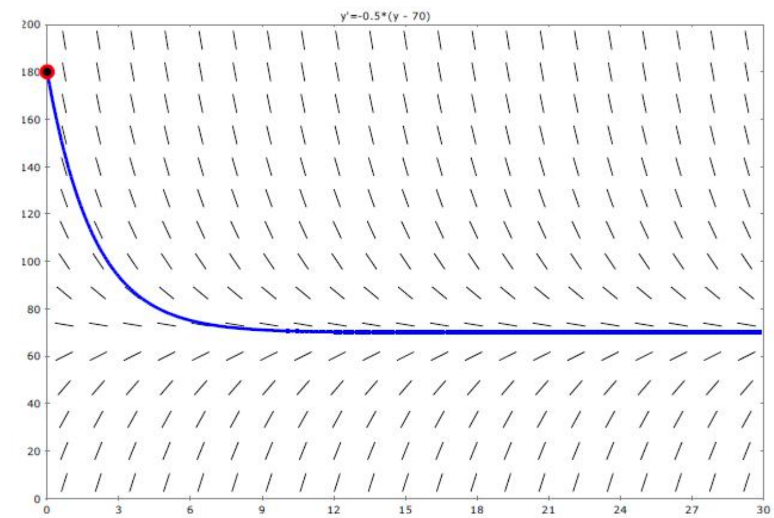
5. If $f(x, y) = xe^{xy}$, then the partial derivatives are
   $\frac{\partial f}{\partial x} = e^{xy} + xye^{xy}$ Note: Product rule, and chain rule in the second term.
   $\frac{\partial f}{\partial y} = x^2e^{xy}$ Note: No product rule, but we did need the chain rule.

Slope Field Examples

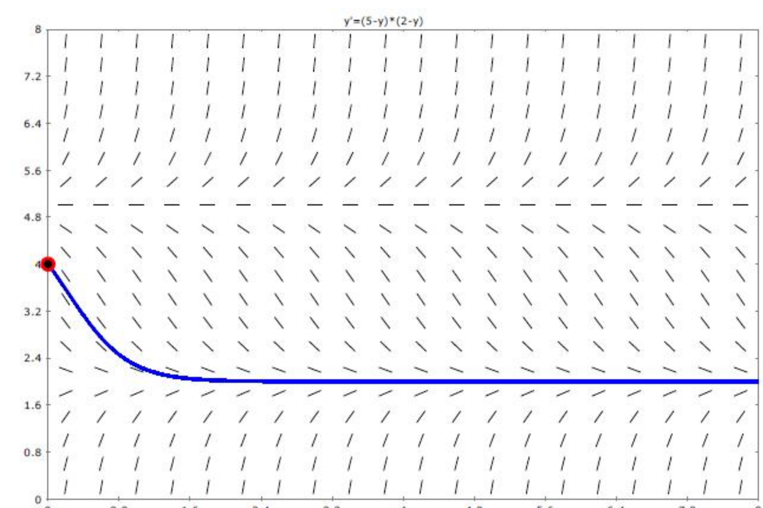
1. $\dfrac{dy}{dt} = 3y$



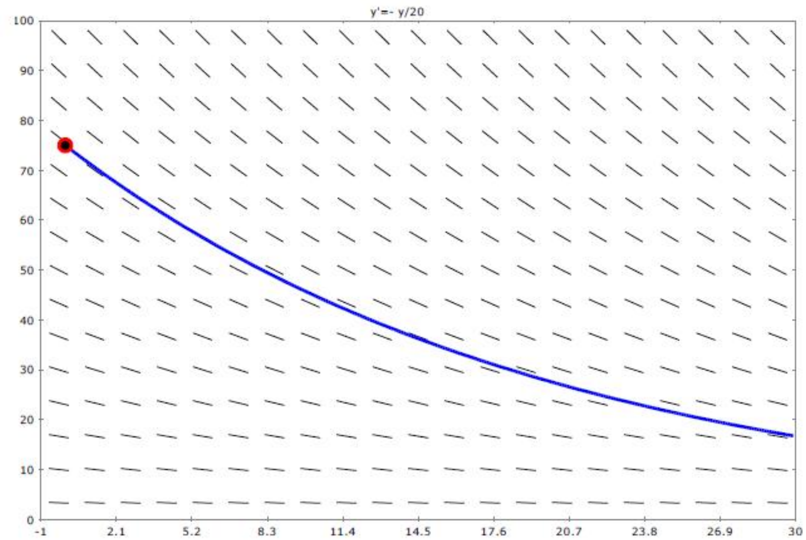
2. $\dfrac{dT}{dt} = -0.5(T - 70)$



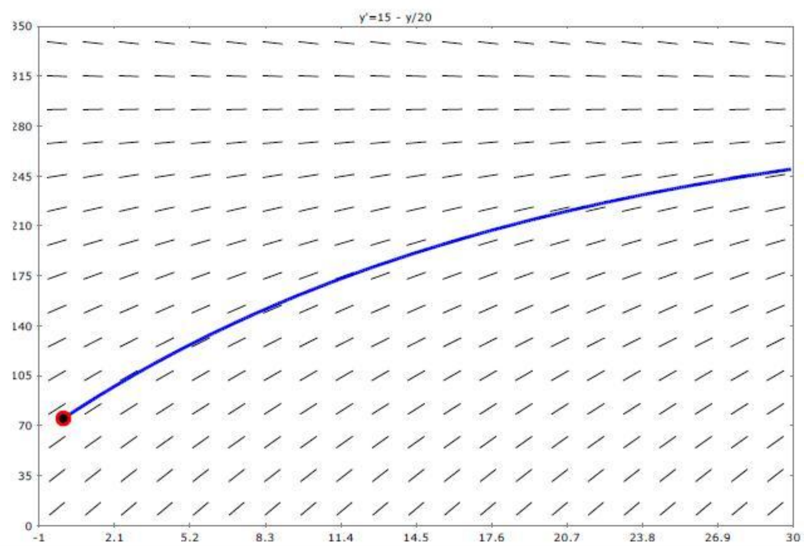
3. $\dfrac{dy}{dt} = (5 - y)(2 - y)$



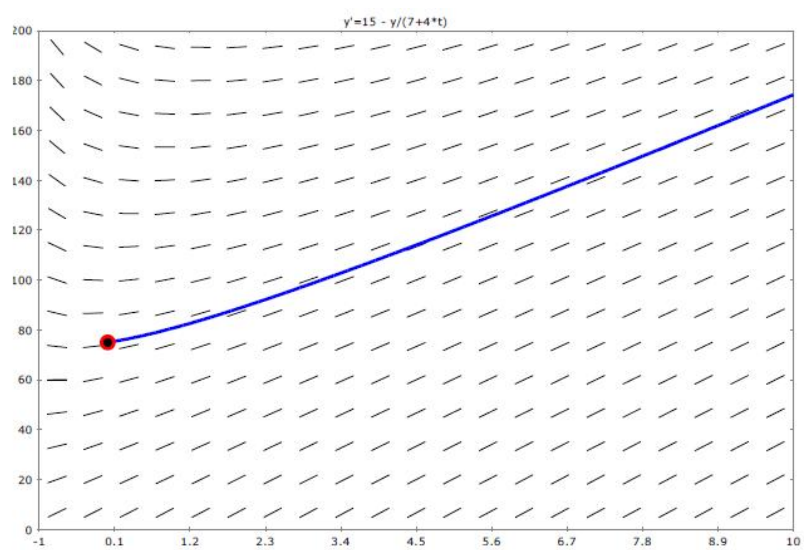
4. $\dfrac{dy}{dt} = -\dfrac{y}{20}$



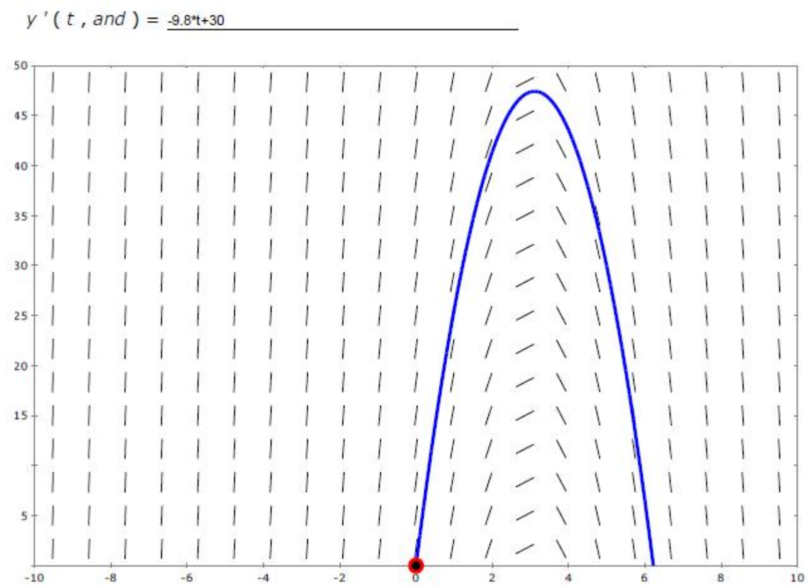
5. $\dfrac{dy}{dt} = 15 - \dfrac{y}{20}$



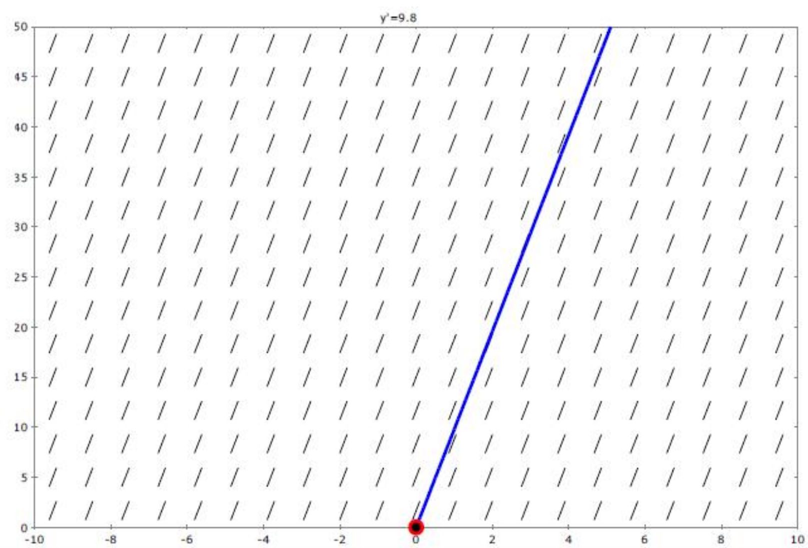
6. $\dfrac{dy}{dt} = 15 - \dfrac{y}{7+4t}$



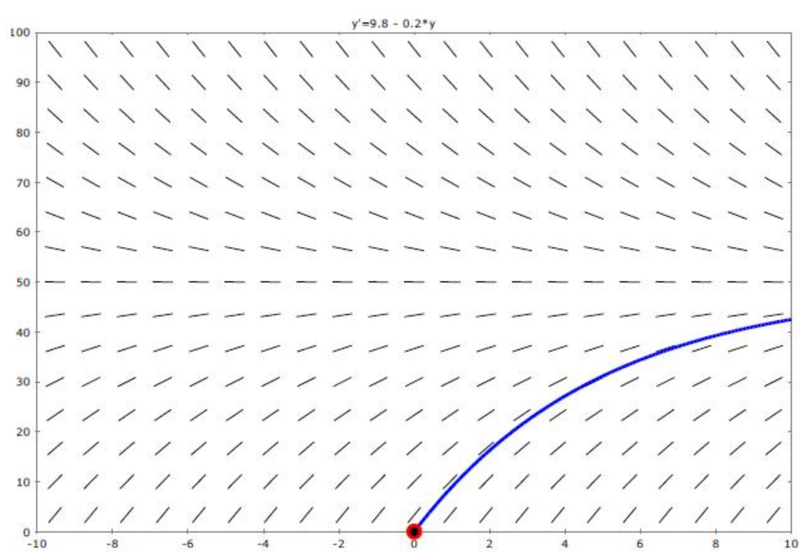
7. $\dfrac{dh}{dt} = -9.8t + 30$



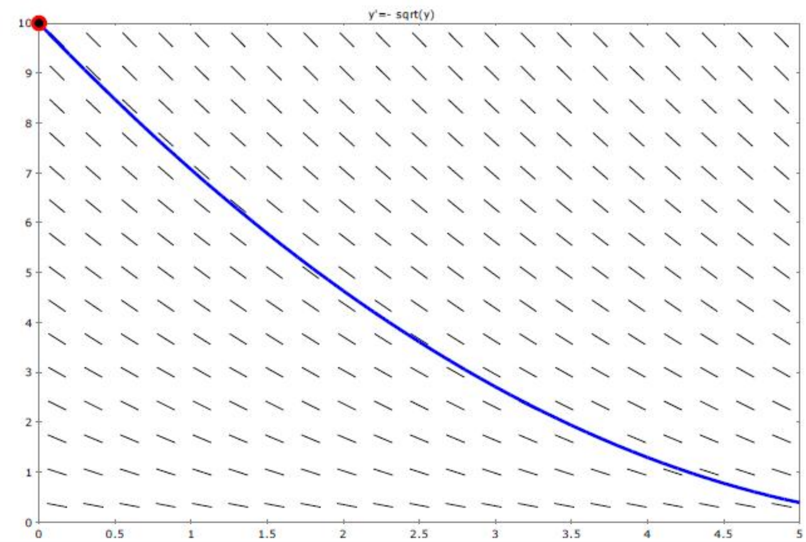
8. $\dfrac{dv}{dt} = 9.8$



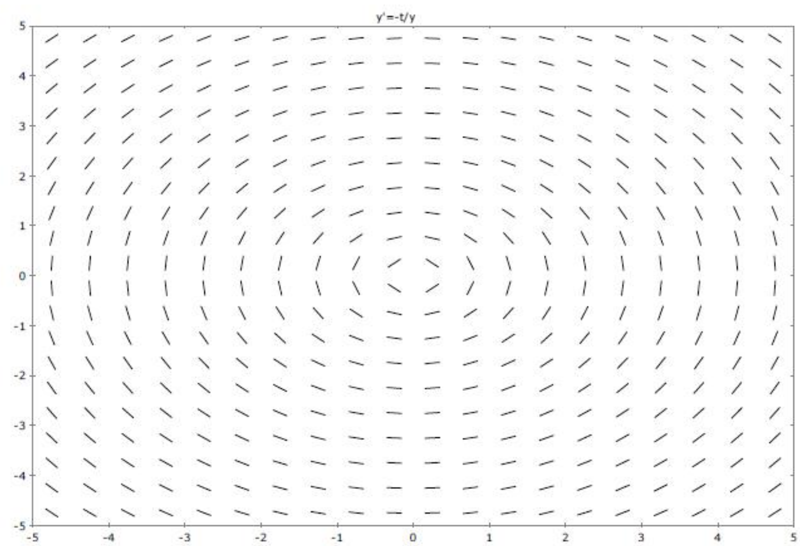
9. $\dfrac{dv}{dt} = 9.8 - 0.2v$



10. $\dfrac{dV}{dt} = -\sqrt{V}$



11. $\dfrac{dy}{dt} = -\dfrac{t}{y}$



12. $\dfrac{dy}{dt} = t^2y^2 - ty^3 + \cos(t)$