MATH 307D
Final Exam Solution
August 23, 2013

Name ______________________________ Student ID #________________

- Your exam should consist of this cover sheet, followed by 5 problems. Check that you have a complete exam.
- Unless otherwise indicated, show all your work and justify your answers.
- Unless otherwise indicated, your answers should be exact values rather than decimal approximations. For example, $\frac{\pi}{4}$ is an exact answer and is preferable to 0.7854.
- You may use a scientific calculator and one double-sided 8.5×11-inch sheet of handwritten notes. All other electronic devices, including graphing or programmable calculators, and calculators which can do calculus, are forbidden.
- The use of headphones, earbuds during the exam is not permitted. Turn off all your electronic devices and put them away.
- If you need more space, write on the back and indicate this. If you still need more space, raise your hand and I'll give you some extra paper to staple onto the back of your test.
- Academic misconduct will guarantee a score of zero on this exam. DO NOT CHEAT.

Problem	Points	S C O R E
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) Solve the initial value problem

$$y'' - 6y' + 9y = t,\ \ y(0) = 0,\ \ y'(0) = 0.$$

Solution: The characteristic equation is $\lambda^2 - 6\lambda + 9 = 0$, with a double root $\lambda = 3$. Therefore, the general solution to the homogeneous equation is

$$C_1 e^{3t} + C_2 t e^{3t}.$$

Let us find a particular solution to the nonhomogeneous equation:

$$y = At + B \quad \Rightarrow \quad y' = A,\ \ y'' = 0.$$

Plug into the equation:

$$-6A + 9(At + B) = t \quad \Rightarrow \quad 9At + 9B - 6A = t \quad \Rightarrow \quad 9A = 1,\ \ 9B - 6A = 0.$$

Solve for A and B:

$$A = 1/9,\ \ B = 2/27.$$

Therefore, the general solution to the nonhomogeneous equation is

$$y(t) = \frac{1}{9}t + \frac{2}{27} + C_1 e^{3t} + C_2 t e^{3t}.$$

We have:

$$y(0) = \frac{2}{27} + C_1 = 0,\ \ y'(0) = \frac{1}{9} + 3C_1 + C_2 = 0.$$

Solve for C_1, C_2:

$$C_1 = -\frac{2}{27},\quad C_2 = -\frac{1}{9} + \frac{2}{9} = \frac{1}{9}.$$

Answer:

$$\boxed{-\frac{2}{27}e^{3t} + \frac{1}{9}te^{3t} + \frac{1}{9}t + \frac{2}{27}}$$

2. (10 points) Find the general solution to the equation

$$t^2 y'' + 3ty' + y = t.$$

Use the fact that $y = t^{-1}$ is a solution to the corresponding homogeneous equation.

Solution: Let $y = C(t)t^{-1}$; then $y' = C't^{-1} - Ct^{-2}$, and $y'' = C''t^{-1} - 2C't^{-2} + 2Ct^{-3}$. Plug this in the equation:

$$t^2\left(C''t^{-1} - 2C't^{-2} + 2Ct^{-3}\right) + 3t\left(C't^{-1} - Ct^{-2}\right) + C(t)t^{-1} = t.$$

Therefore,

$$tC'' - 2C' + 3C' = t \quad \Rightarrow \; tz' + z = t, \; z = C'.$$

Solve for z:

$$tz' + z = 0 \;\Rightarrow\; \frac{z'}{z} = -\frac{1}{t} \;\Rightarrow\; \log|z| = -\log|t| + K \quad \Rightarrow\; |z| = e^K/|t| \quad \Rightarrow\; z = \pm e^K/t.$$

Here, K can take any real values, so $C_1 = \pm e^K$ can take any nonzero values. But we lost the solution $y = 0$, which fits into this scheme by letting $C_1 = 0$. Therefore, $z = C_1/t = C_1 t^{-1}$ is the general solution of $tz' + z = 0$. Let $C_1 = C_1(t)$; then

$$t(-C_1 t^{-2} + C_1' t^{-1}) + C_1 t^{-1} = t \;\Rightarrow\; C_1' = t \;\Rightarrow\; C_1 = \frac{1}{2}t^2 + C_0.$$

Here, C_0 is any real constant. Then

$$z = C_1 t^{-1} = \frac{1}{2}t + C_0 t^{-1}.$$

Therefore,

$$C = \int z(t)dt = \frac{1}{4}t^2 + C_0 \log|t| + C_2,$$

where C_2 is any real constant. Thus,

$$\boxed{z = \frac{1}{4}t + C_0\frac{\log|t|}{t} + C_2\frac{1}{t}}$$

3. (10 points) Using the formula for $L[f''](s)$, find the Laplace transform of $\cos t$.

Solution: Let $f = \cos t$; then $f'' = -\cos t$, and

$$L[f''](s) = -L[f](s).$$

But

$$L[f''](s) = s^2 L[f](s) - sf(0) - f'(0) = s^2 L[f](s) - s.$$

Therefore, we have:

$$-L[f](s) = s^2 L[f](s) - s \quad \Rightarrow \quad L[f](s) = \boxed{\frac{s}{s^2+1}}$$

4. (10 points) Find the general solution to the equation

$$y' = \frac{y^4 - 2y^3 + y^2}{2y - 1}.$$

Solution: We have:

$$\frac{2y-1}{y^4 - 2y^3 + y^2} = \frac{y^2 - (y-1)^2}{y^2(y-1)^2} = \frac{1}{(y-1)}^2 - \frac{1}{y^2}.$$

So after separation of variables we have:

$$\frac{dy}{(y-1)^2} - \frac{dy}{y^2} = dt \quad \Rightarrow \quad \frac{1}{1-y} + \frac{1}{y} = t + C,$$

where C is any real constant. Therefore,

$$\frac{1}{y(1-y)} = t + C \quad \Rightarrow \quad y(1-y) = \frac{1}{t+C} \quad \Rightarrow \quad y^2 - y + \frac{1}{t+C} = 0.$$

We have:

$$y = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{t+C}}\right).$$

But we also lost the equilibrium solutions, which correspond to the y when the right-hand side is equal to zero, that is, when

$$y^4 - 2y^3 + y^2 = 0 \;\Rightarrow\; y^2(y-1)^2 = 0 \;\Rightarrow\; y = 0, 1.$$

Answer:

$$\boxed{y = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{t+C}}\right), \text{ where } C \text{ is a real constant}, \; y = 0, \; y = 1}$$

5. (10 points) Assume we have a lake with volume 1, into which a factory dumps some cyanide, with rate 1 per year. There is a stream going out of the lake, with rate 1 per year. The inflow and outflow have the same rate, but the outflow contains mixed water, and the inflow contains the cyanide with concentration $a(1 + \cos \omega t)$, where $0 < a < 1/2,\ \omega > 0$. Initially, the lake is clean. What is the maximal concentration of cyanide?

Solution: Let $y(t)$ be the quantity of cyanide at time t; then

$$y'(t) = a(1 + \cos \omega t) - y(t), \quad y(0) = 0$$

Solve this: $y'(t) = -y(t) \Rightarrow y(t) = Ce^{-t}$. Then let $C = C(t)$ and plug this into the nonhomogeneous equation. We get:

$$C'(t)e^{-t} - C(t)e^{-t} = a(1 + \cos \omega t) - C(t)e^{-t} \quad \Rightarrow \quad C'(t) = a(1 + \cos \omega t)e^{t} \quad \Rightarrow$$

$$C(t) = \int a(1 + \cos \omega t)e^{t}dt = ae^{t}(1 + A\cos t + B\sin t), \quad A = \frac{1}{\omega^2 + 1}, \quad B = \frac{\omega}{\omega^2 + 1}.$$

Therefore,

$$y(t) = a + aA\cos \omega t + aB\sin \omega t.$$

The maximal value of this function is

$$a + a\sqrt{A^2 + B^2} = \boxed{a + \frac{a}{\sqrt{\omega^2 + 1}}}$$