1. (10 total points) Find the explicit general solutions to the following differential equations. Your answer should be in the form $y = f(t)$, with an undetermined constant appearing somewhere in the equation.

   (a) (5 points)

$$t^2y' + y' = e^y$$

This is a separable equation, so we must separate the variables. Collecting terms we have $(t^2+1)y' = e^y$, so after putting everything on the correct side we get the equation of differentials:

$$e^{-y}\,dy = \frac{1}{t^2+1}\,dt.$$

Integrating on the lift gives us $-e^{-y}$, while integrating on the right yields $\arctan(t) + C$ for some undetermined constant C. Thus, after solving explicitly for y (and relabeling $C \mapsto -C$) we have

$$y = -\log(C - \arctan(t)).$$

   (b) (5 points)

$$ty' - 2y = t^3 - t$$

This is a linear first-order DE, so we must put it in standard form. Dividing through by t gives us

$$y' - \frac{2}{t}y = t^2 - 1.$$

The integrating factor is thus $\mu(t) = e^{\int -\frac{2}{t}\,dt} = e^{-2\ln|t|} = |t|^{-2} = t^{-2}$.
The general solution is then given by

$$\begin{aligned} y &= \frac{1}{\mu(t)}\left(\int \mu(t)(t^2-1)\,dt + C\right) \\ &= t^2\int 1 - t^{-2}\,dt + C\cdot t^2 \\ &= t^2\left(t + \frac{1}{t}\right) + C\cdot t^2 \\ &= t^3 + t + Ct^2. \end{aligned}$$

That is, the general solution the DE is

$$y = t^3 + Ct^2 + t.$$

2. (10 points) Find the solution to the following initial value problem:

$$y'' + y' - 6y = 1 - e^{-t}, \qquad y(0) = 0,\ y'(0) = -1$$

This is a nonhomogeneous second-order linear DE with constant coefficients; we could therefore solve it using the method of undetermined coefficients or with Laplace transforms. We'll use undetermined coefficients, since we use Laplace to solve other DEs in this solution set.

First we obtain the general solution to the homogeneous DE. The characteristic equation for the DE is $r^2 + r - 6 = 0$, which has roots $r = 2$ and $r = -3$. The corresponding general solution to the homogeneous DE is therefore

$$y = c_1 e^{2t} + c_2 e^{-3t}.$$

Now we seek a particular solution to the nonhomogeneous equation. Observe that the forcing function $g(t) = 1 - e^{-t}$ can be be broken into $g_1(t) = 1$ and $g_2(t) = e^{-t}$. Correspondingly the particular solution $Y(t)$ can be written as the sum of the particular solutions to the nonhomogeneous equations $y'' + y' - 6y = 1$ and $y'' + y' - 6y = -e^{-t}$ respectively.

So let $Y_1(t)$ be the particular solution to $y'' + y' - 6y = 1$. Since the forcing function is constant, we guess the particular solution is also constant, i.e. $Y_1 = A$ for constant A. Then $Y_1' = Y_1'' = 0$, so we must have $-6Y_1 = 1$. Hence $Y_1(t) = -\frac{1}{6}$.

Now let $Y_2(t)$ be the particular solution to $y'' + y' - 6y = -e^{-t}$. Since the forcing function is an exponential (which itself is not a solution to the homogeneous DE), we guess the particular solution is also an exponential, i.e. $Y_2(t) = Ae^{-t}$. Then $Y_2' = -Ae^{-t}$ and $Y_2'' = Ae^{-t}$, so

$$Y_2'' + Y_2' - 6Y_2 = Ae^{-t} + (-A)e^{-t} - 6Ae^{-t} = -6Ae^{-t}.$$

Since this must equal the forging function $-e^{-t}$, we must have that $-6A = -1$, i.e. $A = \frac{1}{6}$. Hence $Y_2(t) = \frac{1}{6}e^{-t}$.

We conclude that the particular solution to the full nonhomogeneous DE is

$$Y(t) = Y_1(t) + Y_2(t) = -\frac{1}{6} + \frac{1}{6}e^{-t}.$$

Then the full general solution to the nonhomogeneous DE is the sum of the general solution to the homogeneous DE and the particular solution, so

$$y(t) = c_1 e^{2t} + c_2 e^{-3t} + \frac{1}{6}\left(e^{-t} - 1\right).$$

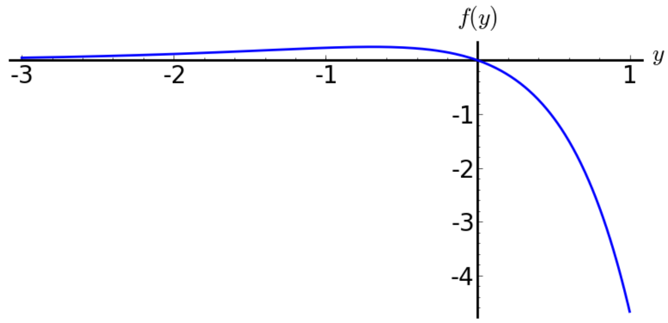
Finally, we apply initial conditions. $y(0) = 0$ implies that $c_1 + c_2 = 0$ i.e. $c_2 = -c_1$, while $y'(0) = -1$ implies that $2c_1 - 3c_2 - \frac{1}{6} = -1$. Solving this system gives us $c_1 = -\frac{1}{6}$ and $c_2 = \frac{1}{6}$. Thus the solution to the IVP is

$$y = -\frac{1}{6}e^{2t} + \frac{1}{6}e^{-3t} + \frac{1}{6}e^{-t} - \frac{1}{6} = \frac{1}{6}\left(-e^{2t} + e^{-t} + e^{-3t} - 1\right).$$

3. (10 total points) Consider the autonomous differential equation

$$y' = e^y - e^{2y},$$

where y is a function of t. Below is a graph of $f(y) = e^y - e^{2y}$ versus y:



(a) (5 points) Find all equilibrium solutions to this differential equation, and classify them according to their stability (stable, unstable or semistable). Be sure to justify your answer.

Recall that equilibrium solutions in an autonomous system are constant solutions, so to find them we must solve for y in the equation

$$f(y) = e^y - e^{2y} = 0.$$

Adding e^{2y} to both sides and dividing by e^y gives us $e^y = 1$; taking logs gives us the unique solution $y = 0$. This corresponds to the sole x-intercept of the graph above.

To ascertain the equilibrium solution's stability, note that $f(y)$ is negative to the left of $y = 0$; this means that y' is negative for any solution that starts out a bit above $y = 0$, so solutions bigger than zero tend to zero. Similarly $f(y) > 0$ for $y < 0$, so solutions that start out a bit below zero increase, and thus head towards the equilibrium point. We conclude that $y = 0$ is a stable equilibrium point. Alternatively we could calculate $\frac{df}{dy}$ at $y = 0$ (or just look at the graph above) and make note that it negative; this corresponds to a stable solution.

(b) (5 points) Suppose we are now looking at the solution to the above DE subject to the initial condition $y(0) = 1$. Use a single step of Euler's method to approximate the value of the solution at $t = 0.5$. You may use decimals in this part of the question, but be sure to maintain at least four digits of precision.

Our stated initial conditions are $t_0 = 0$, $y_0 = 1$. Recall that Euler's method is given by the scheme

$$y_{n+1} = y_n + hf(t_n, y_n)$$

for $n \geq 1$. We only have a single step, so for us $h = \frac{1}{2}$ and $f(t_n, y_n) = e^{y_n} - e^{2y_n}$. Thus we have

$$y_1 = 1 + \frac{1}{2}\left(e^1 - e^2\right) = \frac{2 + e - e^2}{2} \approx -1.3354.$$

Note that this is not a very good approximation, as we know the solution will approach but never cross $y = 0$, since this violates uniqueness.

4. (10 points) My buddy is coming over to watch the game. Unfortunately my fridge has broken down, so I have to resort to alternative measures to cool our drinks down. The drinks are initially at 20 degrees Celsius; one hour before the game starts I place the drinks in an ice box. I note that the rate of cooling of the drinks is proportional to the temperature difference between the drinks and the ice box; moreover, I observe that the proportionality constant is precisely $\frac{1}{50}$ when the units of time are minutes and the units of temperature are degrees Celsius.

   However, the icebox itself is slowing heating up. One hour before the game the icebox is at 0 degrees Celsius, but its temperature is increasing linearly at a rate of 1 degree Celsius every 10 minutes.

   Formulate and solve an initial value problem to find the temperature of the drinks when the game begins.

   Let y be the temperature of the drinks in degrees Celsius as a function of time, and let t be time since I place the drinks in the icebox, (i.e. $t = 0$ is one hour before the game starts), where t is measured in **minutes**. This is the natural choice of units, as the proportionality constant mentioned in the problem statement is given in terms of °C and minutes.

   We know that the drinks are initially at 20 °C, so this gives us the IC $y(0) = 20$. Next, we know that the temperature of the icebox is increasing linearly at a rate of 1 °C every 10 minutes, starting at 0 °C when $t = 0$; hence the temperature of the icebox as a function of time is given by $\frac{t}{10}$.

   Now we invoke Newton's Law of Cooling to set up a differential equation. According to our observations the rate of cooling of the drinks is precisely $\frac{1}{50}$th of the temperature difference between the drinks and the icebox; translating this into mathematics we get the equation

$$\frac{dy}{dt} = -\frac{1}{50}\left(y - \frac{t}{10}\right).$$

   We must be sure to include the minus sign, as if the drinks are warmer than the icebox we expect their temperature to decrease. After multiplying out and shuffling things around we get the first-order initial value problem

$$y' + \frac{1}{50}y = \frac{1}{500}t, \qquad y(0) = 20.$$

   We can solve this linear DE using integrating factors, but Laplace transforms work just as well. Let $y = \phi(t)$ be the solution to this IVP, and let $\Phi(s) = \mathscr{L}[\phi(t)]$. Taking the Laplace transform of both sides of the equation gives us

$$s\Phi - 20 + \frac{1}{50}\Phi = \frac{1}{500} \cdot \frac{1}{s^2},$$

   using $\phi(0) = 20$. Thus, after solving for Φ we get

$$\Phi = \frac{1}{500} \cdot \frac{1}{s^2(s + \frac{1}{50})} + 20 \cdot \frac{1}{(s + \frac{1}{50})}$$

   Using partial fractions in the usual way we find that $\frac{1}{500} \cdot \frac{1}{s^2(s+\frac{1}{50})} = \frac{1}{10} \cdot \frac{1}{s^2} - 5 \cdot \frac{1}{s} + 5 \cdot \frac{1}{(s+\frac{1}{50})}$

[Continued on the next page]

Thus

$$\Phi = \frac{1}{10} \cdot \frac{1}{s^2} - 5 \cdot \frac{1}{s} + 25 \cdot \frac{1}{(s + \frac{1}{50})}.$$

Taking inverse Laplace transforms we find that

$$y = \phi(t) = \frac{t}{10} - 5 + 25e^{-\frac{1}{50}t}.$$

The temperature of the drinks are at the beginning of the game is therefore

$$\phi(60) = \frac{60}{10} - 5 + 25e^{-\frac{60}{50}} = 1 + 25e^{-\frac{6}{5}} \approx 8.53\ ^\circ\text{C}.$$

Cold enough to quench thirst.

5. (10 total points) For the following question you may quote any formula or rule given in the Laplace transform formula sheet at the back of the exam paper.

   (a) (5 points) Compute the Laplace transform of the following function. Your answer should be a function $F(s)$.

$$f(t) = \sin(2t) - 2\cos(t) + t^3 e^{-t}$$

By linearity

$$\mathscr{L}[f] = \mathscr{L}[\sin(2t)] - 2\mathscr{L}[\cos(t)] + \mathscr{L}\left[t^3 e^{-t}\right].$$

From the the formula sheet we see

- $\mathscr{L}[\sin(2t)] = \frac{2}{s^2+4}$,
- $\mathscr{L}[\cos(t)] = \frac{s}{s^2+1}$ and
- $\mathscr{L}\left[t^3 e^{-t}\right] = \mathscr{L}\left[t^3\right]|_(s+1) = \frac{6}{s^4}|_{s+1} = \frac{6}{(s+1)^4}$.

Hence

$$\mathscr{L}\left[\sin(2t) - 2\cos(t) + t^3 e^{-t}\right] = \frac{2}{s^2+4} - \frac{2s}{s^2+1} + \frac{6}{(s+1)^4}.$$

   (b) (5 points) Compute the inverse Laplace transform of the following function. Your answer should be a function $f(t)$.

$$F(s) = \frac{1-e^{-s}}{s^2+s}$$

We see $F(s) = H(s) - e^{-s}H(s)$, where $H(s) = \frac{1}{s^2+s}$. Thus by the rules of inverse Laplace transforms we have

$$f(t) = \mathscr{L}^{-1}[F(s)] = h(t) - u_1(t)h(t-1),$$

where $h(t) = \mathscr{L}^{-1}[H(s)]$.

Thus it remains to compute the inverse Laplace transform of $H(s) = \frac{1}{s^2+s}$. By partial fractions we have that

$$\frac{1}{s^s+s} = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}.$$

Consulting the table we conclude that $\mathscr{L}^{-1}[H(s)] = 1 - e^{-t}$.

Combining this information we get

$$\mathscr{L}^{-1}[F(s)] = 1 - e^{-t} - u_1(t)\left(1 - e^{1-t}\right) = \begin{cases} 1 - e^{-t}, & 0 \le t < 1 \\ (e-1)\cdot e^{-t}, & t \ge 1. \end{cases}$$

6. (10 points) A certain damped oscillator obeys the following differential equation

$$y'' + 6y' + 25y = g(t),$$

where $g(t)$ is an external forcing function. For each of the following possibilities for $g(t)$, write down the form of the steady-state solution. Your answer should be in the form $Y = f(t)$, where f includes undetermined coefficients (A, B, C etc.). **You don't need to compute the actual values of these coefficients in any case**.

Each part is worth 2 points. You don't need to show your working to get full credit for this question.

Note that the characteristic equation of the homogeneous equation is $r^2 + 6r + 25 = 0$ which has roots $r = -3 \pm 4i$, so the general solution to the homogeneous DE is $y = e^{-3t}\left(c_1 \cos(4t) + c_2 \sin(4t)\right)$. So long as the forcing function doesn't contain parts that solve the homogeneous DE i.e. has terms that look like the above for some particular values of c_1 and c_2, we should guess that the particular solution to the nonhomogeneous DE is a general function of the same form as the forcing function.

(a) $g(t) = t^2 + 1$

The forcing function is a quadratic polynomial. None of the terms are a solution to homogeneous DE in any way, we guess a general quadratic polynomial, i.e.

$$Y(t) = At^2 + Bt + C.$$

(b) $g(t) = e^{-t} + \cos(t)$

The forcing function is a combination of an exponential of and a sinusoidal function. Neither of the terms solve the homogeneous DE, so we guess the sum of an exponential and a general sinusoidal, i.e.

$$Y = Ae^{-t} + B\sin(t) + C\cos(t).$$

(c) $g(t) = e^{-t}\cos(t)$

The forcing function is an exponential times a sinusoidal. Note that the coefficients in front of the t's differ from those in the general solution to the homogeneous DE, so this forcing function does not itself solve the homogeneous DE. Thus we can just guess a decaying sinusoidal of the same decay rate and quasi-frequency:

$$Y = e^{-t}\left(A\cos(t) + B\sin(t)\right).$$

[Solution continued on the next page.]

(d) $g(t) = e^{-3t}\sin(4t)$

Similar to above, but now we are in the case of the forcing function itself obeying the homogeneous DE. The rule is thus to take what we would normally guess
(i.e. $Y = e^{-3t}(A\cos(4t) + B\sin(4t))$ and multiply everything by t. Hence our guess should be

$$Y = te^{-3t}(A\cos(4t) + B\sin(4t)).$$

Note that guessing a general linear function times the decaying sinusoidal i.e. something like $Y = e^{-3t}((At+B)\cos(4t) + (Ct+D)\sin(4t))$ would also capture the particular solution; however, because the $e^{-3t}(B\cos(4t) + D\sin(4t))$ part is guaranteed to give your zero when you plug it into the DE, it won't contribute in any way to the solvability of the system of equations in the undetermined coefficients you end up getting. So all such a guess would do is make the algebra more complicated unnecessarily.

(e) $g(t) = te^{-2t}$

The forcing function is a linear polynomial times an exponential. This doesn't obey the homogeneous DE in any way, so we should guess a particular solution of the same form, i.e.

$$Y = (At+B)e^{-2t}.$$

Note that we will need both undetermined constants in this case; guessing $Y = Ate^{-2t}$ won't work - when we differentiate and double differentiate te^{-2t} we get e^{-2t} terms, so we'll need the Be^{-2t} in the guess to be able to account for this.

7. (10 total points) A $\frac{1}{2}$ kg mass is placed on a surface and attached to a horizontal spring with spring constant β kg/s^2, where β is a positive constant. Friction acts on the mass such that when the mass is traveling at 1 m/s it experiences a frictional force of 1 Newton.

   (a) (2 points) Establish a differential equation that the mass obeys.

   Let t be measured in seconds, and let y be the horizontal position of the mass at time t, where y is in meters increasing to the right, and $t = 0$ is some unspecified zero point in time.

   We assume that frictional force is proportional to velocity, acting in the opposite direction. Because the object experiences a force of 1 N when its velocity is 1 m/s, we conclude that the friction coefficient is $\gamma = 1$ kgs^{-1}. There is no forcing function mentioned in the problem outline, so we can assume that forcing is zero. Hence according to the standard setup we have the homogeneous system
   $$\frac{1}{2}y'' + y' + \beta y = 0.$$
   That's it for this part.

   (b) (5 points) For what values of β will the system will be overdamped, critically damped and underdamped respectively? Justify your answer.

   The characteristic equation corresponding to this differential equation is $\frac{1}{2}r^2 + r + \beta = 0$; this has roots
   $$r = \frac{-1 \pm \sqrt{1^2 - 4\cdot\frac{1}{2}\cdot\beta}}{2\cdot\frac{1}{2}} = -1 \pm \sqrt{1-2\beta}.$$
   The nature of solutions to the DE depends on whether the discriminant $1 - 2\beta$ is positive, zero or negative.

   - Overdamping occurs when the CE has real and unequal roots, which corresponds to the discriminant being positive, i.e. $1 - 2\beta > 0$. Since we know $\beta > 0$, we conclude that overdamping occur for $0 < \beta < \frac{1}{2}$.
   - Critical damping occurs when the CE has equal roots, which corresponds to a discriminant of zero, i.e. $1 - 2\beta$. This critical damping occurs for $\beta = \frac{1}{2}$ exactly.
   - Underdamping occurs when the CE has complex roots, which corresponds to a negative discriminant, i.e. $1 - 2\beta < 0$. Thus we get underdamping for $\beta > \frac{1}{2}$.

   (c) (3 points) Find the value of β for which the quasi-frequency of the mass's damped oscillation is exactly 4 radians/sec. [Note: I'm referring to the *angular frequency* ω, not the cyclic frequency.]

   Recall that in a unforced underdamped system, if the roots to the characteristic equation are $r = -\alpha + \omega i$, then a general solution looks like $y = Re^{-\alpha t}\cos(\omega t - \delta)$ for some amplitude R and phase shift δ. Thus we seek β such that solutions to the CE have imaginary part $4i$. But by part (a) the solutions to the CE are $r = -1 \pm \sqrt{1-2\beta} = -1 \pm i\cdot\sqrt{2\beta - 1}$; we thus must have that $\sqrt{2\beta - 1} = 4$. Solving for β gives us the solution $\beta = \frac{17}{2}$.

8. (10 total points + 3 bonus points) Consider the following initial value problem:

$$y'' + 5y' + 6y = g(t), \qquad y(0) = 0, y'(0) = 0,$$

where

$$g(t) = \begin{cases} 0, & 0 \le t < 1 \\ t-1, & 1 \le t < 3 \\ 5-t, & t \ge 3 \end{cases}$$

(a) (2 points) What will the solution be for $0 \le t < 1$? Justify your answer.

The solution will be $y = 0$ for $0 \le t < 1$. This is because the solution can be thought of representing the position of an object in a damped oscillatory system and up until that time there is no forcing taking place. Since the object starts at rest in its equilibrium position it has no energy, and since no energy is being added to the system until $t = 1$ it must remain stationary until then.

(b) (3 points) Rewrite $g(t)$ in terms of Heaviside functions $u_c(t)$. Your answer should be expressible as a linear combination of $u_c(t)$'s each multiplied by some function of t. Make sure to simplify your answer.

We write $g(t)$ in terms of Heaviside functions by building up a series of functions $g_i(t)$ which agree with $g(t)$ for larger and larger time intervals.

The function $g(t)$ is identically zero until $t = 1$, so start with $g_1(t) = 0$; thus $g(t) = g_1(t)$ for $0 \le t < 1$.
Then $g(t)$ becomes $t-1$ at $t = 1$, so let $g_2(t) = g_1(t) + u_1(t)(t-1) = u_1(t)(t-1)$. We then have $g(t) = g_2(t)$ for $0 \le t < 3$.
Finally $g(t)$ becomes $5-t$ for $t \ge 3$, so let $g_3(t) = g_2(t) + u_3(t)[(5-t)-(t-1)]$. Now $g(t) = g_3(t)$ for all $t \ge 0$, so we have found that (after simplifying)

$$g(t) = u_1(t)(t-1) - 2u_3(t)(t-3).$$

(c) (5 points) Let $y = \phi(t)$ be the solution to this IVP. Compute the Laplace transform $\Phi(s)$ of the solution as a function of s. Be sure to simplify your answer.

[NB: you do not need to fully solve the IVP to answer this part of the question.]

We know $y = \phi(t)$ solves the IVP, so we have

$$\phi'' + 5\phi' + 6\phi = u_1(t)(t-1) - 2u_3(t)(t-3)$$

using the result from part (a), with $\phi(0) = \phi'(0) = 0$. Taking Laplace transforms of both sides and invoking linearity we get

$$\begin{aligned} \mathscr{L}[\phi''] + 5\mathscr{L}[\phi'] + 6\mathscr{L}\phi &= \mathscr{L}[u_1(t)(t-1)] - 2\mathscr{L}[u_3(t)(t-3)] \\ \Rightarrow \left(s^2\Phi - s\phi(0) - \phi'(0)\right) + 5\left(s\Phi - \phi(0)\right) + 6\Phi &= \frac{e^{-s}}{s^2} - 2\frac{e^{-3s}}{s^2} \\ \Rightarrow (s^2+5s+6)\Phi &= \frac{e^{-s} - 2e^{-3s}}{s^2} \end{aligned}$$

using the fact that $\phi(0) = \phi'(0) = 0$. Thus by dividing by $s^2 + 5s + 6 = (s+2)(s+3)$ we solve for Φ in terms of s to get

$$\Phi(s) = \frac{e^{-s} - 2e^{-3s}}{s^2(s+2)(s+3)}.$$

(d) (3 bonus points) Find the solution $y = \phi(t)$ to the IVP.

Let $H(s) = \frac{1}{s^2(s+2)(s+3)}$ and $h(t) = \mathscr{L}^{-1}(s)$. We note that $\Phi(s) = e^{-s}H(s) - 2e^{-3s}H(s)$, so we know then that

$$\phi(t) = u_1(t)h(t-1) - 2u_3(t)h(t-3).$$

It thus remains to compute the inverse Laplace transform of $H(s)$.

Now by partial fractions $\frac{1}{s^2(s+2)(s+3)} = \frac{As+B}{s^2} + \frac{C}{s+2} + \frac{D}{s+3}$. Clearing denominators we get

$$1 = (As+B)(s+2)(s+3) + Cs^2(s+3) + Ds^2(s+2).$$

Evaluating at $s = -2$ gives us $4C = 1$, so $C = \frac{1}{4}$.
Evaluating at $s = -3$ gives us $-9D = 1$, so $D = -\frac{1}{9}$.
Evaluating at $s = 0$ gives us $6B = 1$, so $B = \frac{1}{6}$.
Finally, evaluating at, say, $s = 1$ gives us $1 = 12(A+B) + 4C + 3D = 12A + 2 + 1 - \frac{1}{3}$; after collecting terms and solving for A we get $A = -\frac{5}{36}$.
Hence $H(s) = -\frac{5}{36} \cdot \frac{1}{s} + \frac{1}{6} \cdot \frac{1}{s^2} + \frac{1}{4} \cdot \frac{1}{s+2} - \frac{1}{9} \cdot \frac{1}{s+3}$. Thus

$$h(t) = \mathscr{L}^{-1}H(s) = -\frac{5}{36} + \frac{t}{6} + \frac{1}{4}e^{-2t} - \frac{1}{9}e^{-3t} = \frac{1}{36}\left(-5 + 6t + 9e^{-2t} - 4e^{-3t}\right).$$

We now have a complete description of the solution to the IVP in terms of Heaviside functions and shifts of $h(t)$, so this completes the answer.

Table of Laplace Transforms

In this table, n always represents a positive integer, and a and c are real constants.

$f(t) = \mathscr{L}^{-1}[F(s)]$	$F(s) = \mathscr{L}(f(t))$	
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
t^n, n a positive integer	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^n e^{ct}$, n a positive integer	$\frac{n!}{(s-c)^{n+1}}$	$s > c$
t^a, $a > -1$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cosh(at)$	$\frac{s}{s^2-a^2}$	$s > \|a\|$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$s > \|a\|$
$e^{ct}\cos(at)$	$\frac{s-c}{(s-c)^2+a^2}$	$s > c$
$e^{ct}\sin(at)$	$\frac{a}{(s-c)^2+a^2}$	$s > c$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$e^{ct}f(t)$	$F(s-c)$	
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$	$c > 0$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	
$t^nf(t)$	$(-1)^nF^{(n)}(s)$	