Math 307, Final Exam Name: ______________________

Winter 2013

Instructions.

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question. Come back to the question you left if you have time at the end.
- There are 10 questions on 10 pages. Make sure your exam is complete.
- You are allowed one double-sided sheet of notes in your own handwriting. You may not use someone else's note sheet. You may use a simple scientific calculator, but you don't need to. No fancy calculators or other electronic devices allowed. If you didn't bring a simple calculator, then just don't use a calculator.
- Simplify trig functions from the unit circle (e.g. $\sin(0) = 0$), and also simplify expressions like $e^{\ln t}$.
- **Show your work**, unless instructed otherwise. If you need more space, raise your hand and I'll give you some extra paper to staple onto the back of your test.
- Don't cheat. If I see that you aren't following the rules, I will report you to UW.

Question	Points	Score
1	12	
2	8	
3	12	
4	12	
5	6	
6	10	
7	10	
8	14	
9	12	
10	4	
Total:	100	

1. Find the general form of the solution for the following differential equations.

(a) (5 points)

$$\frac{dy}{dt} = (te^t)/(2y+1)$$

Solution: Separate the variables: $2y+1\,dy = te^t\,dt$. Then integrate both sides, using integration by parts on the right:

$$y^2 + y = te^t - e^t + C.$$

Finally, solve for y. You can complete the square or use the quadratic formula.

$$\begin{aligned} y &= \frac{-1 \pm \sqrt{1-4(-te^t - e^t - C)}}{2} \\ &= \frac{-1 \pm \sqrt{1+4(te^t + e^t + C)}}{2} \end{aligned}$$

(b) (7 points)

$$u'' + u' + 3 = t$$

Solution: Oops! Typo. I meant to write $3u$, not 3. As written, the answer is $u = \frac{1}{2}t^2 - 4t + Ce^{-t} + D$.

Here's the solution I intended:

The roots of the characteristic equation are $-\frac{1}{2} \pm \frac{\sqrt{11}}{2}i$, so the general form of the homogeneous solution is

$$c_1 e^{-t/2} \cos\left(\frac{\sqrt{11}}{2}t\right) + c_2 e^{-t/2} \sin\left(\frac{\sqrt{11}}{2}t\right)$$

Now we need a particular solution $U(t)$, of the form $U = At + B$. Then $U' = A$ and $U'' = 0$. Plugging them in, we get

$$A + 3At + 3B = t,$$

so $A + 3B = 0$ and $3A = 1$. Thus $A = \frac{1}{3}$ and $B = -\frac{1}{9}$. So the general solution is

$$c_1 e^{-t/2} \cos\left(\frac{\sqrt{11}}{2}t\right) + c_2 e^{-t/2} \sin\left(\frac{\sqrt{11}}{2}t\right) + \frac{1}{3}t - \frac{1}{9}.$$

2. (8 points) Consider the nonhomogeneous differential equation

$$y'' + y' - 2y = f(t)$$

for some function $f(t) \neq 0$. For each value of $f(t)$ below, tell me what form the **particular solution** $Y(t)$ of the differential equation would take. (For example, you might write "$A\cos(t) + Bt$", if you thought a solution would take that form.) You don't need to tell me what the solution actually is! You may leave the coefficients undetermined. Use capital letters A, B, C, and so on for the coefficients.

Remember that sometimes we need to multiply by extra powers of t. For full credit, you need to write all the extra t's that are required.

No justification necessary. Two points each.

$f(t)$	Form of particular solution $Y(t)$:
$\cos(t) + \cos(2t)$	$A\cos(t) + B\sin(t) + C\cos(2t) + D\sin(2t)$
$e^t + \sin(t)$	$Ate^t + B\sin(t) + C\cos(t)$
$t\cos(4t)$	$At\cos(4t) + Bt\sin(4t) + C\cos(4t) + D\sin(4t)$
$6t^2 - t$	$At^2 + Bt + C$

3. (12 points) A small tropical island has a population of butterflies. With no predators, this population would triple every year (increasing at a rate proportional to the current population). However, the island also has frogs that eat 400 butterflies every year. Assume there are 1000 butterflies on the island now. In four years, how many butterflies will be on the island?

Solution: Our equation has the form

$$\frac{dP}{dt} = rP - 400.$$

First we need to find r. If there were no predators, the butterfly population would be modeled by the DE

$$\frac{dP}{dt} = rP,$$

which has solution $P = P_0 e^{rt}$, if P_0 is the initial population. The population triples every year, so after one year $P = 3P_0$. Then we have

$$3P_0 = P_0 e^{r \cdot 1}$$

so we solve for r and get $r = \ln 3$. Now our equation is

$$\frac{dP}{dt} = (\ln 3)P - 400.$$

The integrating factor is then $e^{(-\ln 3)t}$. Integrating both sides and solving for P, we get

$$P = \frac{400}{\ln 3} + Ce^{(\ln 3)t}.$$

Using that $P(0) = 1000$, we get $C \approx 635.9$. Finally, plugging in $t = 4$, we find that there will be about $51,872$ butterflies on the island.

4. Consider the function

$$f(t) = \begin{cases} t-1 & 0 \le t < 3 \\ 2 & 3 \le t < 5 \\ t^2 - 23 & t \ge 5 \end{cases}$$

(a) (2 points) Is $f(t)$ continuous? Explain.

Solution: Yes. It is piecewise continuous, so we just need to check that the definitions at the endpoints match. They do: $3 - 1 = 2$, and $2 = (5)^2 - 23$.

(b) (3 points) Write $f(t)$ in terms of the unit step functions $u_c(t)$.

Solution:

$$\begin{aligned} &(t-1)(1-u_3(t)) + 2(u_3(t) - u_5(t)) + (t^2 - 23)u_5(t) \\ =& t - 1 + (3-t)u_3(t) + (t^2 - 25)u_5(t) \end{aligned}$$

(c) (7 points) Find the Laplace transform $\mathcal{L}\{f(t)\}$.

Solution:

$$\begin{aligned} \mathcal{L}\{f\} &= \mathcal{L}\{t-1\} + \mathcal{L}\{u_3(t)(3-t)\} + \mathcal{L}\{u_5(t)(t^2-25)\} \\ &= \frac{1}{s^2} - \frac{1}{s} + e^{-3s}\mathcal{L}\{-t\} + e^{-5s}\mathcal{L}\{(t+5)^2 - 25\} \\ &= \frac{1}{s^2} - \frac{1}{s} + e^{-3s}\frac{-1}{s^2} + e^{-5s}\mathcal{L}\{t^2 + 10t\} \\ &= \frac{1}{s^2} - \frac{1}{s} - \frac{e^{-3s}}{s^2} + e^{-5s}\left(\frac{2}{s^3} + \frac{10}{s^2}\right). \end{aligned}$$

5. You've learned a few things about Laplace transforms and how they behave with other operations. For example, you know that $\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$, and you know you can factor out constants.

   In this problem, I'm going to give you two **FALSE** rules about how Laplace transforms work. You should provide examples of functions to prove these rules wrong.

   (a) (3 points) False rule # 1:

   $$\mathcal{L}\{f(t) \cdot g(t)\} = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}$$

   Find two functions $f(t), g(t)$ for which this equation isn't true. Check carefully, and show your work.

   Solution: Many possibilities here. One easy one is $f(t) = 1, g(t) = t$. Then we have $\mathcal{L}\{fg\} = \mathcal{L}\{t\} = \frac{1}{s^2}$, but $\mathcal{L}\{f\}\mathcal{L}\{g\} = \frac{1}{s} \cdot \frac{1}{s^2} = \frac{1}{s^3}$.

   (b) (3 points) False rule # 2:

   $$\mathcal{L}\left\{\frac{d}{dt}f(t)\right\} = \frac{d}{ds}\left(\mathcal{L}\{f(t)\}\right).$$

   Write down a function $f(t)$ for which this equation isn't true. Again, check carefully and show your work.

   Solution: Again, many possibilities. You could set $f(t) = 1$, then $f' = 0$ so $\mathcal{L}\{f'\} = 0$. On the other hand, $\mathcal{L}\{1\} = \frac{1}{s}$, and this does not differentiate to 0 with respect to s!

6. Find the inverse Laplace transform of the functions below.

(a) (5 points)

$$\frac{e^{-3s}}{(s-1)^2(s-3)}$$

Solution: Use partial fractions first:

$$\frac{1}{(s-1)^2(s-3)} = \frac{-1/2}{(s-1)^2} + \frac{-1/4}{s-1} + \frac{1/4}{s-3},$$

where the first and last came from the cover-up method, and the middle one can be found by plugging in $s = 0$ (or any other value). Now take inverse $\mathcal{L}$ of that whole expression:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2(s-3)}\right\} = -\frac{1}{2}(te^t) - \frac{1}{4}e^t + \frac{1}{4}e^{3t}.$$

Finally, we need to plug $t-3$ into that expression, and multiply by $u_3(t)$. Final answer:

$$u_3(t)\left(-\frac{1}{2}(te^{t-3}) - \frac{1}{4}e^{t-3} + \frac{1}{4}e^{3(t-3)}\right).$$

(b) (5 points)

$$\frac{2s-4}{s^2+2s+5}$$

Solution: Since the denominator is irreducible, we'll complete the square:

$$\frac{2s-4}{(s+1)^2+4}.$$

Rearrange the numerator and break up the expression:

$$\frac{2(s+1)}{(s+1)^2+4} + \frac{-6}{(s+1)^2+4}$$

Factor out 2 from the numerator on the left and -3 from the numerator on the right:

$$2\left(\frac{s+1}{(s+1)^2+4}\right) - 3\left(\frac{2}{(s+1)^2+4}\right)$$

Finally, take $\mathcal{L}^{-1}$ of each term using the table:

$$2e^{-t}\cos(2t) - 3e^{-t}\sin(2t).$$

7. A spring-mass system has a spring constant of 20 N/m. A mass of 4 kg is attached to the end of the spring. Let γ be the damping constant of the system

   (a) (2 points) What is the natural frequency of the system?

   Solution: We just need the roots of $4r^2 + 20 = 0$, or $r^2 + 5 = 0$. The roots are $\pm\sqrt{5}i$, so the natural frequency is $\sqrt{5}$.

   (b) (2 points) Suppose $\gamma = 0.003$ and the system has an applied external force $F(t) = \cos(\omega t)$, where ω is the number you found in part (a). You would like to get the biggest possible amplitude for the steady state response. Should you increase the frequency of F, decrease it, or keep it the same? Explain.

   Solution: Decrease it a little. The resonant frequency is slightly smaller than the natural frequency when γ is small.

   (c) (2 points) Now suppose that the amplitude of the steady state response for the system in part (b) is as big as possible. Is this amplitude bigger or smaller than the amplidude of F (which is 1)? Explain.

   Solution: Bigger: this is the resonance effect with small γ.

   (d) (2 points) For which value of γ is the system **critically damped**? Call this number γ_d.

   Solution: The system is critically damped if the characteristic equation has only one real root. This happens when $\gamma^2 - 4(4)(20) = 0$, or $\gamma = 8\sqrt{5}$.

   (e) (2 points) Write down (but don't solve) an initial value problem for this spring-mass system, with $\gamma = \gamma_d$, and an external force of $2\cos(2t)$. The mass starts from rest at equilibrium.

   Solution: The IVP is $4u'' + 8\sqrt{5}u' + 20u = 2\cos(2t)$, $y(0) = 0$, $y'(0) = 0$.

8. (14 points) Let $u(t)$ describe the position of a mass on a spring. Suppose u satisfies the differential equation

$$u'' + 2u' + u = \cos(2t)$$

with $u(0) = 2$, $u'(0) = 0$. Find a formula for $u(t)$ and **put a box around the steady-state part of the solution.**

Solution: The characteristic equation has only one root — -1 — so the general form of the homogeneous solution is

$$c_1 e^{-t} + c_2 t e^{-t}.$$

We need to find a particular solution. It will have the form $U = A\cos(2t) + B\sin(2t)$. Differentiate this expression:

$$U' = -2A\sin(2t) + 2B\cos(2t)$$
$$U'' = -4A\cos(2t) - 4B\sin(2t)$$

Now plug these into the DE and simplify. We get

$$(-3A - 4B)\cos(2t) + (4A - 3B)\sin(2t) = \cos(2t),$$

so $-3A - 4B = 1$ and $4A - 3B = 0$. Solving the system, we get $A = \frac{-3}{25}, B = \frac{4}{25}$. Thus the general form of the solution is

$$u(t) = c_1 e^{-t} + c_2 t e^{-t} - \frac{3}{25}\cos(2t) + \frac{4}{25}\sin 2t.$$

Now we just need to plug in the initial conditions. Since $u(0) = 2$, we get an equation $2 = c_1 - \frac{3}{25}$. Differentiating u and plugging in $t = 0$, we get the equation $0 = -c_1 + c_2 + \frac{8}{25}$. Solving the system, we find $c_1 = \frac{53}{25}$ and $c_2 = -\frac{45}{25} = \frac{9}{5}$. So the final answer is

$$u = \frac{53}{25}e^{-t} + \frac{9}{5}te^{-t} - \frac{3}{25}\cos(2t) - \frac{4}{25}\sin(2t)$$

and the steady-state part is the last two terms.

9. (12 points) Solve the initial value problem.

$$y'' - 3y' + 2y = \begin{cases} 1 & 0 \le t < 1 \\ 0 & t \ge 1, \end{cases}$$

where $y(0) = 0$, $y'(0) = 0$.

Solution: Write in terms of the unit step function:

$$y'' - 3y' + 2y = 1 - u_1(t).$$

Now take $\mathcal{L}$ of both sides, and set $Y = \mathcal{L}\{y\}$:

$$s^2Y - 3sY + 2Y = \frac{1}{s} - \frac{e^{-s}}{s}.$$

(The initial conditions being zero simplify this expression a lot.) Solve for Y and factor the denominators:

$$Y = \frac{1}{s(s-1)(s-2)} - e^{-s}\left(\frac{1}{s(s-1)(s-2)}\right)$$

Expand the rational function with partial fractions:

$$\frac{1}{s(s-1)(s-2)} = \frac{1/2}{s} + \frac{-1}{s-1} + \frac{2}{s-2}$$

Now undo $\mathcal{L}$:

$$y = \mathcal{L}^{-1}\{Y\} = \frac{1}{2} - e^t + 2e^{2t} + u_1(t)\left(\frac{1}{2} - e^{t-1} + 2e^{2(t-1)}\right).$$

10. (4 points) Consider the nonseparable, nonlinear differential equation

$$\frac{dy}{dt} = 10y^2 - 2t$$

with initial condition $y(0) = 1$. Use Euler's method with step size $h = 0.1$ to estimate $y(0.2)$. You may use the table below if it helps you, but you don't need to. **Clearly label your answer!**

(x, y)	$f(x, y)$	$h \cdot f(x, y)$	$y + h \cdot f(x, y)$	$x + h$
$(0, 1)$	10	1	2	0.1
$(0.1, 2)$	39.8	3.98	5.98	0.2

Solution: $y(0.2) \simeq 5.98$.

Congratulations! You are done. Have a great spring break!