1. A 32-lb object is attached to a (giant) spring, stretching it by 8 ft. Assume that when the object is traveling at 3 ft/s, it experiences a damping force of 15 lb. There is also an external force of $F(t) = 10\cos 2t + 10\sin 2t$ ft/s acting on the object.

At time $t = 0$, you pull the object 1 ft downward, and release it with initial velocity 1 ft/s downward.

(a) Find the amplitude and phase of the steady-state solution. (You may include square roots and trigonometric functions in your answer.)

(b) Find the position of the object as a function of time.

Setup: $mu'' + \gamma u' + ku = F(t)$

- $m = \dfrac{32\text{ lb}}{32\text{ ft/s}^2} = 1 \ \dfrac{\text{lb}\cdot\text{s}^2}{\text{ft}}$
- $\gamma u' = 15\text{ lb}$ when $u' = 3\ \frac{\text{ft}}{\text{s}} \Rightarrow \gamma = \dfrac{15\text{ lb}}{3\text{ ft/s}} = 5\ \dfrac{\text{lb}\cdot\text{s}}{\text{ft}}$
- $k = \dfrac{mg}{L} = \dfrac{32\text{ lb}}{8\text{ ft}} = 4\ \dfrac{\text{lb}}{\text{ft}}$

$\Rightarrow$ $u'' + 5u' + 4u = 10\cos 2t + 10\sin 2t$
$u(0) = 1$
$u'(0) = 1$

(a) Steady-state solution is the particular solution from the method of undetermined coefficients

Homogeneous solution: $y_c(t) = c_1 e^{-t} + c_2 e^{-4t}$ $\leftarrow$ characteristic equation: $r^2 + 5r + 4 = 0$
$(r+1)(r+4) = 0$
$r = -1, -4$

Template: $Y(t) = A\cos 2t + B\sin 2t$
(neither $\cos 2t$ nor $\sin 2t$ are solutions of homogeneous equation)

Plug in: $Y(t) = A\cos 2t + B\sin 2t$
$Y'(t) = 2B\cos 2t - 2A\sin 2t$
$Y''(t) = -4A\cos 2t - 4B\sin 2t$

$Y'' + 5Y' + 4Y = (\cancel{-4A} + 10B \cancel{+4A})\cos 2t + (\cancel{-4B} - 10A \cancel{+4B})\sin 2t$
$= 10B\cos 2t - 10A\sin 2t$

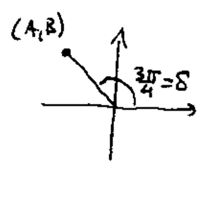
$10\cos 2t + 10\sin 2t = 10B\cos 2t - 10A\sin 2t$

$\Rightarrow B = 1,\ A = -1$: $Y(t) = -\cos 2t + \sin 2t$

Amplitude: $R = \sqrt{A^2 + B^2} = \boxed{\sqrt{2}}$

Phase: $\tan\delta = \dfrac{B}{A} = -1$ $\rightarrow$ $\delta = \tan^{-1}(-1) + \pi = \boxed{\dfrac{3\pi}{4}}$

$(A,B) = (-1,1)$ in left halfplane $(x<0)$



(b) General solution: $Y(t) + y_c(t) = y(t) = -\cos 2t + \sin 2t + c_1 e^{-t} + c_2 e^{-4t}$

Solve for c_1, c_2: $1 = y(0) = -1 + c_1 + c_2$
$y'(t) = 2\sin 2t + 2\cos 2t - c_1 e^{-t} - 4c_2 e^{-4t}$
$1 = y'(0) = 2 - c_1 - 4c_2$

$\Rightarrow c_1 = \frac{7}{3},\ c_2 = -\frac{1}{3}$

$$\boxed{y(t) = -\cos 2t + \sin 2t + \underbrace{\tfrac{7}{3}e^{-t} - \tfrac{1}{3}e^{-4t}}_{\text{transient solution}}}$$

2. A 1kg mass is attached to a spring. The spring constant is $k = 25\text{kg/s}^2$, but you don't know the damping coefficient γ. If the quasiperiod is $2\pi/3$, find γ.

quasiperiod $= \frac{2\pi}{3} = \frac{2\pi}{\mu}$ $\Rightarrow$ quasifrequency is $\mu = 3$

Know $\mu = \frac{\sqrt{4mk-\gamma^2}}{2m} = \frac{\sqrt{4\cdot 1\cdot 25-\gamma^2}}{2\cdot 1} = \frac{\sqrt{100-\gamma^2}}{2}$

$\Rightarrow \frac{\sqrt{100-\gamma^2}}{2} = 3$

$\Rightarrow \sqrt{100-\gamma^2} = 6$

$\Rightarrow \boxed{\gamma = 8}$

3. All critically damped systems have the same Q factor. Find this Q factor.

$Q = \frac{\sqrt{mk}}{\gamma}$

For a critically damped system, $\gamma^2 - 4mk = 0$ $\Rightarrow \gamma^2 = 4mk$

$\Rightarrow \gamma = 2\sqrt{mk}$

$\Rightarrow Q = \frac{\cancel{\sqrt{mk}}}{2\cancel{\sqrt{mk}}} = \boxed{\frac{1}{2}}$

4. Find the general solution to the ODE

$$y'' - 6y' + 9y = te^{3t} + e^{-t}.$$

Method of undetermined coefficients.

Homogeneous solution: Characteristic equation: $r^2 - 6r + 9 = 0$

$$(r-3)^2 = 0$$

$$r = 3 \quad \text{repeated root}$$

$$y_c(t) = c_1 e^{3t} + c_2 te^{3t}$$

Particular solution:

te^{3t} $\quad e^{-t}$ $\quad g(t)$

$\cancel{3te^{3t}}$ $\quad e^{3t}$ $\quad \cancel{e^{-t}}$ $\quad g'(t)$

$\cancel{3e^{3t}}$ $\quad g''(t)$

First try: $Y(t) = Ate^{3t} + Be^{3t} + Ce^{-t}$

e^{3t}, te^{3t} are solutions to the homogeneous equation, though, so multiply by t

$$Y(t) = \cancel{Ate^{3t}} + \cancel{Be^{3t}} + Ce^{-t}$$

$\cancel{At^2e^{3t}}$ $\quad Bte^{3t}$ ← te^{3t} still sol'n to homog. equation

At^3e^{3t} $\quad Bt^2e^{3t}$ ← t^3e^{3t}, t^2e^{3t} aren't solutions to homogeneous equation, so stop.

$$Y(t) = At^3e^{3t} + Bt^2e^{3t} + Ce^{-t}$$

$$Y'(t) = 3At^3e^{3t} + 3At^2e^{3t} + 3Bt^2e^{3t} + 2Bte^{3t} - Ce^{-t}$$

$$= 3At^3e^{3t} + (3A+3B)t^2e^{3t} + 2Bte^{3t} - Ce^{-t}$$

$$Y''(t) = 9At^3e^{3t} + 9At^2e^{3t} + (9A+9B)t^2e^{3t} + (6A+6B)te^{3t} + 6Bte^{3t} + 2Be^{3t} + Ce^{-t}$$

$$\begin{aligned}
Y'' - 6Y' + 9Y &= \cancel{(9A - 6(3A) + 9A)}\, t^3e^{3t} \\
&\quad + \cancel{(18A + 9B - 6(3A+3B))}\, t^2e^{3t} \\
&\quad\quad +6B \\
&\quad + (6A + 12B - 12B)\, te^{3t} \\
&\quad + 2B\, e^{3t} \\
&\quad + (C + 6C + 9C)\, e^{-t} \\
&= 6Ate^{3t} + 2Be^{3t} + 16Ce^{-t}
\end{aligned}$$

$$te^{3t} + e^{-t} = 6Ate^{3t} + 2Be^{3t} + 16Ce^{-t}$$

$$\Rightarrow 6A = 1, \quad 2B = 0, \quad 16C = 1$$

$$A = \tfrac{1}{6}, \quad B = 0, \quad C = \tfrac{1}{16}$$

particular solution: $Y(t) = \frac{1}{6}t^3e^{3t} + \frac{1}{16}e^{-t}$

General solution: $\boxed{y(t) = Y(t) + y_c(t) = \frac{1}{6}t^3e^{3t} + \frac{1}{16}e^{-t} + c_1e^{3t} + c_2te^{3t}}$

5. Given that $y_1(t) = t$ is a solution, find another solution to the ODE

$$t^2y'' - t(t+2)y' + (t+2)y = 0$$

that is not a multiple of t. What is the general solution?

Reduction of order:

$$y_2(t) = v(t)y_1(t) \qquad = tv$$
$$y_2' = v'y_1 + vy_1' \qquad = tv' + v$$
$$y_2'' = v''y_1 + 2v'y_1' + vy_1'' \qquad = tv'' + 2v'$$

Plug in y_2:

$$0 = t^2y_2'' - t(t+2)y_2' + (t+2)y_2$$
$$= t^2(tv''+2v') - t(t+2)(tv'+v) + (t+2)tv$$
$$= t^3v'' + \cancel{2t^2v'} - t^3v' - \cancel{2t^2v'} - \cancel{t^2v} - \cancel{2tv} + \cancel{t^2v} + \cancel{2tv}$$
$$= t^3(v''-v')$$

Divide both sides by t^3:

$$0 = v'' - v'$$
$$v' = v''$$
$$1 = \frac{v''}{v'}$$
$$\int 1\,dt = \int \frac{v''}{v'}\,dt$$
$$t + c = \ln|v'|$$
$$ce^t = |v'|$$
$$ce^t = v'$$
$$\int ce^t = \int v'$$
$$ce^t + d = v$$

So $y_2(t) = v(t)y_1(t) = c\,\underline{te^t} + d\underline{t}$

This is actually the general solution –

One answer for a solution that is not a multiple of t is $y_2 = te^t$

General solution: $\boxed{cte^t + dt}$