• Complete all questions.
• You may use a scientific calculator during this examination. Other electronic devices (e.g. cell phones) are not allowed, and should be turned off for the duration of the exam.
• You may use one hand-written 8.5 by 11 inch page of notes.
• Show all work for full credit.
• You have 60+ minutes to complete the exam.
1. Find the general solution to the differential equations:

(a) (5 points)

\[ y'' - 2y' - 3y = te^t. \]

The characteristic equation is \( r^2 - 2r - 3 = (r - 3)(r + 1) = 0 \), so \( r = 3, -1 \). Then the homogeneous solution is \( y_h = c_1 e^{3t} + c_2 e^{-t} \).

We guess a particular solution: \( y_p = (a_0 + a_1 t) e^t \). Then \( y'_p = [(a_0 + a_1) + a_1 t] e^t \) and \( y''_p = [(a_0 + 2a_1) + a_1 t] e^t \). Plugging this into the left-hand side, we get

\[
    e^t [(a_0 + 2a_1 - 2(a_0 + a_1) - 3a_0) + (a_1 - 2a_1 - 3a_1) t] = te^t.
\]

So that we have two equations:

\[-4a_1 = 1 \quad \text{and} \quad -4a_0 = 0.\]

So \( y_p = -\frac{t}{4} e^t \), and

\[ y(t) = y_h + y_p = c_1 e^{3t} + c_2 e^{-t} - \frac{t}{4} e^t. \]

(b) (5 points)

\[ y'' - 2y' - 3y = g(t) \]

Hint: Express your answer using integrals.

Using the homogeneous solution above, we have \( y_1(t) = e^{3t} \) and \( y_2(t) = e^{-t} \). The variation of parameters formula tells us that \( y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \) where

\[
    u_1(t) = -\int_0^t y_2(s) \frac{g(s)}{W(y_1,y_2)(s)} ds,
\]

and

\[
    u_2(t) = \int_0^t y_1(s) \frac{g(s)}{W(y_1,y_2)(s)} ds.
\]

We compute the wronskian:

\[
    W(e^{3t}, e^{-t})(t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 3e^{3t} & -e^{-t} \end{vmatrix} = -4e^{2t}.
\]

Then

\[
    u_1(t) = -\int_0^t e^{-s} \frac{g(s)}{-4e^{2s}} ds = \frac{1}{4} \int_0^t e^{-3s} g(s) ds,
\]

\[
    u_2(t) = \int_0^t e^{3s} \frac{g(s)}{-4e^{2s}} ds = -\frac{1}{4} \int_0^t e^{s} g(s) ds,
\]

and so \( y(t) = y_h(t) + y_p(t) \) gives

\[
    y(t) = c_1 e^{3t} + c_2 e^{-t} + \frac{e^{3t}}{4} \int_0^t e^{-3s} g(s) ds - \frac{e^{-t}}{4} \int_0^t e^{s} g(s) ds.
\]
2. (10 points) Suppose that the motion of a spring-mass system satisfies

\[ u'' + u' + 1.5u = \sin(2t) \]

and that the mass starts \((t = 0)\) at the equilibrium position from rest. Find the position \(u(t)\) at any time \(t\).

We first solve the homogeneous equation. We get \(r^2 + r + 1.5 = 0\), so \(r = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}i\). Then \(e^{rt} = e^{-t/2}(\cos(t\sqrt{5}/2) + i\sin(t\sqrt{5}/2))\); taking real and imaginary parts we get

\[ u_h(t) = c_1e^{-t/2}\cos(t\sqrt{5}/2) + c_2e^{-t/2}\sin(t\sqrt{5}/2). \]

Then we need to find a particular solution. We might as well use undetermined coefficients. So we replace \(\sin(2t)\) with \(e^{2ti}\), since \(Im(e^{2ti}) = \sin(2t)\). Then we guess that \(v_p(t) = Ae^{2ti}\). Plugging into the equation, we get:

\[ e^{2ti}(-4 + 2i + 1.5)A = e^{2ti}. \]

This tells us that \(A = \frac{1}{-2.5+2i} = \frac{-2.5-2i}{41/4} = -\frac{10}{41} - \frac{8}{41}i\). Then

\[ v_p(t) = -\left(\frac{10}{41} + \frac{8}{41}i\right)e^{2ti} = -\left(\frac{10}{41} + \frac{8}{41}i\right)(\cos(2t) + i\sin(2t)) \]

\[ = \left(\frac{10}{41}\cos(2t) - \frac{8}{41}\sin(2t)\right) + i\left(-\frac{8}{41}\cos(2t) + \frac{10}{41}\sin(2t)\right). \]

Thus \(u_p(t) = Im(v_p(t)) = -\left(\frac{8}{41}\cos(2t) + \frac{10}{41}\sin(2t)\right)\) is a particular solution to the original problem. We have then that

\[ u(t) = u_h(t) + u_p(t) = c_1e^{-t/2}\cos(t\sqrt{5}/2) + c_2e^{-t/2}\sin(t\sqrt{5}/2) + \left(\frac{8}{41}\cos(2t) + \frac{10}{41}\sin(2t)\right). \]

Since the initial conditions are \(u(0) = 0 = u'(0)\), we get that

\[ 0 = u(0) = c_1 - \frac{8}{41} \]
\[ 0 = u'(0) = -\frac{c_1}{2} + \frac{\sqrt{5}}{2} + \frac{20}{41}. \]

Then \(c_1 = \frac{8}{41}\), and so \(c_2 = \frac{12}{41} \cdot \frac{2}{\sqrt{5}} = \frac{24\sqrt{5}}{205}\). Finally,

\[ u(t) = \frac{8}{41}e^{-t/2}\cos(t\sqrt{5}/2) + \frac{24\sqrt{5}}{205}e^{-t/2}\sin(t\sqrt{5}/2) + \left(\frac{8}{41}\cos(2t) + \frac{10}{41}\sin(2t)\right). \]
3. Compute the following Laplace transforms using the definition, or using only the num-
bers (1),(13),(14),(18), and (19) on the table.

(a) (5 points)

\[ \mathcal{L}\{t^2e^{\pi t}\} \]

We know from (1) that \( \mathcal{L}\{1\} = \frac{1}{s} \). Then from (19), we know that
\( \mathcal{L}\{t^2 \cdot 1\} = (\frac{1}{s})'' = \frac{2}{s^2} \). Finally, from (14), we know that
\( \mathcal{L}\{e^{\pi t}t^2\} = \frac{2}{(s-\pi)^2} \).

(b) (5 points)

\[ \mathcal{L}\{u_3(t)(t^2 - 2t - 1)\} \]

If \( g(t) = t^2 - 2t - 1 \), then \( f(t) := g(t+3) = (t+3)^2 - 2(t+3) - 1 = t^2 + 6t + 9 - 2t - 6 - 1 = t^2 + 4t + 2 \). Observe that \( f(t-3) = g(t) \), so that \( \mathcal{L}\{u_3(t)g(t)\} = \mathcal{L}\{u_3(t)f(t-3)\} \) which is \( e^{-3t}\mathcal{L}\{f(t)\} \) by (13).

We use linearity to compute \( \mathcal{L}\{f(t)\} \): since we computed \( \mathcal{L}\{t\} = \frac{1}{s} \) and \( \mathcal{L}\{t^2\} = \frac{2}{s^3} \) above, we only need to know \( \mathcal{L}\{t\} \). For this we can use (18): \( \mathcal{L}\{2t\} = \mathcal{L}\{(t^2)'.\} = s\mathcal{L}\{t^2\} = \frac{2}{s^2} \). Dividing both sides by 2 (and using linearity), we get that \( \mathcal{L}\{t\} = \frac{1}{s^2} \).

Now by linearity: \( \mathcal{L}\{f(t)\} = \frac{2}{s^3} + \frac{4}{s^2} + \frac{2}{s} \), so that
\[
\mathcal{L}\{u_3(t)g(t)\} = e^{-3s} \left( \frac{2}{s^3} + \frac{4}{s^2} + \frac{2}{s} \right) = \frac{e^{-3s}}{s^3} \left( 2s^2 + 4s + 2 \right).
\]
4. (10 points) Use the Laplace transform to solve the following IVP using the table:

\[ y'' - y = \begin{cases} 
1 & t < 2 \\
\frac{t}{3} & 2 \leq t 
\end{cases} \]

\[ y(0) = 0 \quad y'(0) = 0. \]

We first re-write the driving function \( g(t) \) (right-hand side) using the unit step functions. We get that \( g(t) = 1 + u_2(t)\left(\frac{t}{3} - 1\right) \). Then we take the Laplace transform of both sides:

\[ \mathcal{L}\{y''\} - \mathcal{L}\{y\} = \frac{1}{s} + \mathcal{L}\{u_2(t)\left(\frac{t}{3} - 1\right)\}. \]

Use (13) to evaluate the last Laplace transform: first, observe that \( \frac{t}{3} - 1 = \frac{t - 2}{3} - \frac{1}{3} \), so if \( f(t) = \frac{t}{3} - \frac{1}{3} \), we have that

\[ \mathcal{L}\{u_2(t)\left(\frac{t}{3} - 1\right)\} = \mathcal{L}\{u_2(t)f(t - 2)\} = e^{-2s}\left(\frac{1}{3s^2} - \frac{1}{3s}\right). \]

For the left-hand side, we use the usual formulas (18 on the table):

\[ (s^2 - 1)\mathcal{L}\{y\} = \frac{1}{s} + \frac{e^{-2s}}{3s^2} - \frac{e^{-2s}}{3s}. \]

Solve: \( \mathcal{L}\{y\} = \frac{-1 + \frac{1}{2} e^{-t} + \frac{1}{2} e^t}{s(s + 1)(s - 1)}. \)

Partial fractions expansion gives \( \frac{1}{s(s + 1)(s - 1)} = \frac{-1}{s} + \frac{1/2}{s + 1} + \frac{1/2}{s - 1} \). Thus:

\[ \mathcal{L}^{-1}\left\{\frac{1}{s(s + 1)(s - 1)}\right\} = -1 + \frac{1}{2} e^{-t} + \frac{1}{2} e^t. \]

Partial fractions expansion gives \( F(s) := \frac{1}{s^2(s + 1)(s - 1)} = \frac{A}{s} + \frac{-1}{s^2} + \frac{-1/2}{s + 1} + \frac{1/2}{s - 1} \) if we just use the cover-up method. To solve for \( A \), just pick a different value for \( s \) (other than \( s = -1, 0, 1 \)); we pick \( s = 2 \), and get that \( A = 0 \). So it is easy now to compute

\[ \mathcal{L}^{-1}\{F(s)\} = -t - \frac{1}{2} e^{-t} + \frac{1}{2} e^t =: f(t). \]

It follows that \( \mathcal{L}^{-1}\left\{\frac{e^{-2s}F(s)}{s}\right\} = \frac{u_2(t)}{3} f(t - 2) \) by (13); then finally,

\[ y(t) = -1 + \frac{1}{2} e^{-t} + \frac{1}{2} e^t - \frac{u_2(t)}{3} \left(-1 + \frac{1}{2} e^{-(t-2)} + \frac{1}{2} e^{t-2}\right) + \frac{u_2(t)}{3} \left(-(t-2) - \frac{1}{2} e^{t-2} + \frac{1}{2} e^{t-2}\right) \]

\[ = -1 + \cosh(t) + \frac{u_2(t)}{3} (3 - t - \cosh(t - 2) + \sinh(t - 2)). \]
5. (10 points) A spring-mass system has a spring constant of 2N/m. A mass of 8kg is attached to the spring. Let $\gamma$ be the damping constant of the system.

(a) (2 points) What is the natural frequency of the system?

$$w_0 = \sqrt{k/m} = \sqrt{2/8} = 1/2.$$ 

(b) (2 points) Suppose $\gamma = 9$. Is the (free) system under-damped, over-damped or critically damped?

The discriminant $\Delta = \gamma^2 - 4mk = 9^2 - 4(8)(2) > 0$. Thus the system is over-damped.

(c) (2 points) From now on, suppose $\gamma = 2$. Find the quasi-frequency of the (free) system.

$$\mu = w_0 \sqrt{1 - \frac{\gamma^2}{4km}} = \frac{1}{2} \sqrt{1 - \frac{4}{4(8)(2)}} = \frac{1}{2} \sqrt{15/16}.$$ 

(d) (2 points) Suppose we apply an external force $F(t) = 5 \cos(\omega t)$ N. What is the resonant frequency of this forced system?

$$w_{res} = w_0 \sqrt{1 - \frac{\gamma^2}{2mk}} = \frac{1}{2} \sqrt{1 - \frac{4}{2(8)(2)}} = \frac{1}{2} \sqrt{7/8}.$$ 

(e) (2 points) Write down the initial value problem corresponding to this forced system where $w$ is the resonant frequency, and the mass starts at rest from the equilibrium position.

$$8u'' + 2u' + 2u = 5 \cos \left( t \frac{\sqrt{7/8}}{2} \right) \quad \begin{cases} u(0) = 0 \\ u'(0) = 0. \end{cases}$$
6. (3 bonus points) Compute the laplace transform of \( \ln(t) \) by following these steps.

(a) (1 point) Differentiate the formula

\[
\mathcal{L}(t^p) = \int_0^\infty e^{-st}t^p \, dt = \frac{\Gamma(p + 1)}{s^{p+1}}
\]

with respect to \( p \). For the the middle term, move the differential operator \( \frac{d}{dp} \) inside the integral and apply it to the integrand.

\[
\frac{\Gamma'(p + 1)s^{p+1} - \Gamma(p + 1)s^{p+1}\ln(s)}{s^{2p+2}} = \frac{d}{dp} \left( \frac{\Gamma(p + 1)}{s^{p+1}} \right) = \frac{d}{dp} \int_0^\infty e^{-st}t^p \, dt
\]

\[
= \int_0^\infty e^{-st} \frac{d}{dp} (t^p) \, dt
\]

\[
= \int_0^\infty e^{-st}t^p \ln(t) \, dt.
\]

(b) (1 point) Simplify as much as possible, and then evaluate the resulting expression at \( p = 0 \).

If we simplify the left-hand side, we get

\[
\frac{\Gamma'(p + 1) - \Gamma(p + 1)\ln(s)}{s^{p+1}} = \int_0^\infty e^{-st}t^p \ln(t) \, dt.
\]

Evaluating at \( p = 0 \), we get

\[
\frac{\Gamma'(1) - \ln(s)}{s} = \int_0^\infty e^{-st} \ln(t) \, dt.
\]

(c) (1 point) What is \( \mathcal{L}(\ln(t)) \)?

By definition, \( \mathcal{L}\{\ln(t)\} = \int_0^\infty e^{-st} \ln(t) \, dt \), which is

\[
\frac{\Gamma'(1) - \ln(s)}{s}
\]

by the previous formula.
Table of Laplace transforms:

<table>
<thead>
<tr>
<th>f(t) = \mathcal{L}^{-1}{F(s)}</th>
<th>F(s) = \mathcal{L}{f(t)}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 1</td>
<td>\frac{1}{s^n}, \ s &gt; 0</td>
</tr>
<tr>
<td>2. \ e^{at}</td>
<td>\frac{1}{s-a}, \ s &gt; a</td>
</tr>
<tr>
<td>3. \ t^n, \ n = \text{positive integer}</td>
<td>\frac{n!}{s^{n+1}}, \ s &gt; 0</td>
</tr>
<tr>
<td>4. \ t^p, \ p &gt; -1</td>
<td>\frac{\Gamma(p+1)}{s^{p+1}}, \ s &gt; 0</td>
</tr>
<tr>
<td>5. \ \sin at</td>
<td>\frac{a}{s^2+a^2}, \ s &gt; 0</td>
</tr>
<tr>
<td>6. \ \cos at</td>
<td>\frac{s}{s^2+a^2}, \ s &gt; 0</td>
</tr>
<tr>
<td>7. \ \sinh at</td>
<td>\frac{a}{s^2-a^2}, \ s &gt;</td>
</tr>
<tr>
<td>8. \ \cosh at</td>
<td>\frac{s}{s^2-a^2}, \ s &gt;</td>
</tr>
<tr>
<td>9. \ \ e^{at} \sin bt</td>
<td>\frac{b}{(s-a)^2+b^2}, \ s &gt; a</td>
</tr>
<tr>
<td>10. \ \ e^{at} \cos bt</td>
<td>\frac{s-a}{(s-a)^2+b^2}, \ s &gt; a</td>
</tr>
<tr>
<td>11. \ \ t^n \ e^{at}, \ n = \text{positive integer}</td>
<td>\frac{n!}{(s-a)^{n+1}}</td>
</tr>
<tr>
<td>12. \ \ u_c(t)</td>
<td>\frac{e^{-cs}}{s}, \ s &gt; 0</td>
</tr>
<tr>
<td>13. \ \ u_c(t) f(t-c)</td>
<td>\ e^{-cs} \ F(s)</td>
</tr>
<tr>
<td>14. \ \ e^{ct} \ f(t)</td>
<td>\ F(s-c)</td>
</tr>
<tr>
<td>15. \ \ f(ct)</td>
<td>\frac{1}{c} F\left(\frac{s}{c}\right), \ c &gt; 0</td>
</tr>
<tr>
<td>16. \ \ \int_0^t f(t-\tau) g(\tau) d\tau</td>
<td>\ F(s) G(s)</td>
</tr>
<tr>
<td>17. \ \ \delta(t-c)</td>
<td>\ e^{-cs}</td>
</tr>
<tr>
<td>18. \ \ f^{(n)}(t)</td>
<td>\ s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)</td>
</tr>
<tr>
<td>19. \ \ (-t)^n f(t)</td>
<td>\ F^{(n)}(s)</td>
</tr>
</tbody>
</table>