Skills Review: Solving Two-by-Two Systems

In this course, you will often have to solve a two-by-two system of linear equations that looks like

$$ax_1 + bx_2 = P;$$
$$cx_1 + dx_2 = Q,$$

where a, b, c, d, P, and Q are all numbers and you are solving for x_1 and x_2. Here is a reminder of your goals and your tools for solving such equations.

1. The goal is to **combine** the two equations into one equation that has only one variable so that you can solve for that variable.
2. Your two main combining tools:
   - Add or Subtract the two equations from each other. This is valid because if you add equal things to equal things you get equal things! Note that you can also multiply or divide both sides of any equation by a number (in order to set up a situation where adding/subtracting will lead to cancellation).
   - Substitute! Solve for one variable in the first equation and substitute into the second.
3. Once you have solved for one variable, you can substitute back into one (or both) of the original equations to find the other variable. As a check on your work, you should plug into both equations.

Basic example: Solve the system $\begin{array}{ll} \text{(i)} & 2x_1 + x_2 = 5 \\ \text{(ii)} & x_1 - x_2 = 4 \end{array}$

- *Solution 1: Combining by Adding/Subtracting*
  Notice the cancellation that will happen if we add!
  Adding corresponding sides of (i) and (ii) gives a combined equation of $3x_1 = 9$. Thus, $x_1 = 3$.
  Substituting back into (i) gives $2(3) + x_2 = 5$, so $x_2 = -1$.
  Substituting back into (ii) gives $(3) - x_2 = 4$, so $x_2 = -1$.
  Thus, the only solution is $x_1 = 3$ and $x_2 = -1$.
- *Solution 2: Substituting*
  Solving for x_2 in the first equation, we can rewrite equation (i) as $x_2 = 5 - 2x_1$.
  Substituting into (ii), we get a combined equation of $x_1 - (5 - 2x_1) = 4$ which simplifies to $3x_1 - 5 = 4$. Solving gives $3x_1 = 9$, so $x_1 = 3$.
  Substituting back into our simplified version of (i) gives $x_2 = 5 - 2(3) = -1$.
  Substituting back into (ii) gives $(3) - x_2 = 4$, so $x_2 = -1$.
  Thus, the only solution is $x_1 = 3$ and $x_2 = -1$.

The first method is sometimes faster, but it requires some cleverness. The second method always takes the same amount of time and requires no cleverness. That's it, now you can solve linear 2-by-2 systems!

Here is another one to try on your own:

Example: Solve the system $\begin{array}{ll} \text{(i)} & 2x_1 + 2x_2 = 6 \\ \text{(ii)} & 3x_1 - x_2 = 2 \end{array}$

Comments about the solution: You can either start by dividing the first equation by 2, then adding. Or just solve for x_1 or x_2 in the first equation and substituting into the second. Both will work. The answer you should get is $x_1 = \frac{5}{4}$ and $x_2 = \frac{7}{4}$.

Some very important theoretic comments about two-by-two systems
There are three things that can happen in a two-by-two system:

1. UNIQUE solution: The most 'likely' situation (*i.e.* if you randomly pick numbers for coefficients you probably get a system with a unique solution). See two examples on the last page.

2. NO solution: Happens if the 'left-hand side' of the second equation is a multiple of the first, but the 'right-hand side' is not the same multiple. For example: $\begin{array}{ll} \text{(i)} & x_1 - 2x_2 = 10; \\ \text{(ii)} & 3x_1 - 6x_2 = 50. \end{array}$ In this example, (i) $x_1 - 2x_2 = 10$ and (ii) $3(x_1 - 2x_2) = 50$ can't happen because 50 is NOT 3 times 10. There is NO solution.

3. INFINITELY many solutions: This happens if both sides are the same multiple of each other. For example: $\begin{array}{ll} \text{(i)} & x_1 - 2x_2 = 10; \\ \text{(ii)} & 3x_1 - 6x_2 = 30. \end{array}$ Notice that both sides of equation (ii) are exactly 3 times equation (i). In fact, equations (i) and (ii) are two different ways to write the exact same equation. Thus, all solutions will satisfy $x_1 = 10 + 2x_2$. For example, one solution is $x_1 = 10$, $x_2 = 0$, another is $x_1 = 12$, $x_2 = 1$, another is $x_1 = 14$, $x_2 = 2$, and so on ...

The Determinant:
For a system of the form $\begin{array}{l} ax_1 + bx_2 = P; \\ cx_1 + dx_2 = Q, \end{array}$, we define the two-by-two **determinant** by

$$\text{determinant} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Note: For a two-by-two system if the **determinant is zero**, then the 'left-hand sides' are multiples of each other. For example, the system $\begin{array}{ll} \text{(i)} & x_1 - 2x_2 = 10 \\ \text{(ii)} & 3x_1 - 6x_2 = 30 \end{array}$ has a determinant of $(1)(-6) - (-2)(3) = 0$.

Existence and Uniqueness Theorem for Linear Systems:
From what we have already said, we can summarize

1. if $ad - bc \neq 0$, then the system has a **unique solution**.

2. if $ad - bc = 0$, then the system will have no solution or infinitely many solutions (depending on the values of P and Q).

Cramer's Rule: (Just for your interest, not required)
If you combined and solved the general system $\begin{array}{l} ax_1 + bx_2 = P; \\ cx_1 + dx_2 = Q, \end{array}$ you would find that if there is a unique answer then it is always is equal to

$$x_1 = \frac{Pd - bQ}{ad - bc} = \frac{\begin{vmatrix} P & b \\ Q & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \quad x_2 = \frac{aQ - Pc}{ad - bc} = \frac{\begin{vmatrix} a & P \\ c & Q \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

You can use Cramer's rule to solve if you wish, but it is usually just as fast to combine and solve. To learn facts about larger systems (3-by-3 and 4-by-4), then you have to take a course in linear algebra (Math 308).