6.3: Step Functions

One of the main applications of Laplace transform is discontinuous forcing (which we will explore in section 6.4). But first we need to learn about how to work with discontinuous functions in an organized way.

Definition: We define the **unit step function**, or **Heaviside function**, by

$$u_c(t) = \begin{cases} 0, & t < c; \\ 1, & t \geq c. \end{cases}$$

We will see that many discontinuous function can be written in terms of $u_c(t)$.

Key Theorems:

1. $\mathcal{L}\{u_c(t)\} = \dfrac{e^{-cs}}{s}$
2. If $F(s) = \mathcal{L}\{f(t)\}$, then $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s)$.

   For example, $\mathcal{L}\{u_2(t)e^{3(t-2)}\} = e^{-2s}\mathcal{L}\{e^{3t}\} = \frac{e^{-2s}}{s-3}$
3. If $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)\mathcal{L}^{-1}\{F(s)\}(t-c) = u_c(t)f(t-c)$.

   For example, $\mathcal{L}^{-1}\{\frac{e^{-2s}}{s-3}\} = u_2(t)\mathcal{L}^{-1}\{\frac{1}{s-3}\}(t-2) = u_2(t)e^{3(t-2)}$.

Putting Step Functions Together:
To add/subtract step functions, we simply add/subtract corresponding overlapping formulas. Here is an example:

$$f(t) = u_2(t) + 4u_3(t) - 2u_{10}(t) = \begin{cases} (0) + 4(0) - 2(0) = 0, & t < 2; \\ (1) + 4(0) - 2(0) = 1, & 2 \leq t < 3; \\ (1) + 4(1) - 2(0) = 5, & 3 \leq t < 10; \\ (1) + 4(1) - 2(1) = 3, & t \geq 10. \end{cases}$$

Here is another example:

$$h(t) = 15u_3(t) - 15u_5(t) = \begin{cases} 15(0) - 15(0) = 0, & t < 3; \\ 15(1) - 15(0) = 15, & 3 \leq t < 5; \\ 15(1) - 15(1) = 0, & t \geq 5. \end{cases}$$

Thus, we see that we can get many other piecewise functions by simply using combinations of step functions.

Constructing Piecewise Functions:
Assume you want to build a function that has a value of zero for the first 10 seconds, and then it looks like the Cosine functions (starting at the peak) from 10 seconds and onward. How can we do this? Here are two useful tricks:

1. $f(t-c)$ = 'the same as $f(t)$ but shifted c units to the right'
   So for example, $\cos(t-10)$ will look just like the Cosine wave starting at the peak when $t=10$.

2. $u_c(t)f(t-c)$ = 'the value is zero until $t=c$, then it looks like $f(t)$ shifted c units to the right'.
   For example,
   $$u_{10}(t)\cos(t-10) = \begin{cases} (0)\cos(t-10) = 0, & t < 10; \\ (1)\cos(t-10) = \cos(t-10), & t \geq 10. \end{cases}$$
   Here is another example:
   $$3t + u_5(t)(t-5)^3 = \begin{cases} 3t + (0)(t-5)^3 = 3t, & t < 5; \\ 3t + (1)(t-5)^3 = 3t + (t-5)^3, & t \geq 5. \end{cases}$$

Writing a Given Piecewise Function In Terms of Step Functions:
Finally, to apply our methods and theorems, we need to be able to write a given piecewise function in terms of step functions. We can do this by adding or subtracting appropriately at each jump. Consider the two problems:

1. Write $f(t)$ in terms of step functions, where $f(t) = \begin{cases} 4, & 0 \leq t < 1; \\ 7, & 1 \leq t < 5; \\ 1, & 5 \leq t < 11; \\ 0, & t \geq 11. \end{cases}$

   We will start with $t = 0$ and work our way through each discontinuity:

   (a) The function starts with $f(t) = 4$ as the correct rule for $0 \leq t < 1$.

   (b) At $t = 1$ the function 'jumps up 3' to get to 7,
   so the value is given by $f(t) = 4 + 3u_1(t)$ for $0 \leq t < 5$.

   (c) At $t = 5$ the function 'jumps down 6' to get to 1,
   so the value is given by $f(t) = 4 + 3u_1(t) - 6u_5(t)$ for $0 \leq t < 11$.

   (d) At $t = 11$ the function 'jumps down 1' to get to 0,
   so the value is given by $f(t) = 4 + 3u_1(t) - 6u_5(t) - u_{11}(t)$ for all values of t.

   Thus, this function can be written as $f(t) = 4 + 3u_1(t) - 6u_5(t) - u_{11}(t)$.

2. Write $f(t)$ in terms of step functions, where $f(t) = \begin{cases} t, & 0 \leq t < 2; \\ 3, & 2 \leq t < 6; \\ 8 - t, & 6 \leq t < 10; \\ 2, & t \geq 10. \end{cases}$

   Again, we start with $t = 0$ and work our way through each discontinuity:

   (a) The function starts with $f(t) = t$ as the correct rule for $0 \leq t < 2$.

   (b) At $t = 2$ the function changes from t to 3, so we need to subtract t and add 3.
   The value is given by $f(t) = t + (3 - t)u_2(t)$ for $0 \leq t < 6$.

   (c) At $t = 6$ the function changes from 3 to $8 - t$, so we need to subtract 3 and add $8 - t$.
   The value is given by $f(t) = t + (3 - t)u_2(t) + (8 - t - 3)u_6(t) = t + (3 - t)u_2(t) + (5 - t)u_6(t)$ for $0 \leq t < 10$.

   (d) At $t = 10$ the function changes from $8 - t$ to 2, so we need to subtract $8 - t$ and add 2.
   The value is given by $f(t) = t + (3 - t)u_2(t) + (5 - t)u_6(t) + (2 - 8 + t)u_{10}(t)$
   $= t + (3 - t)u_2(t) + (5 - t)u_6(t) + (t - 6)u_{10}(t)$ for all values of t.

   Thus, this function can be written as $f(t) = t + (3 - t)u_2(t) + (5 - t)u_6(t) + (t - 6)u_{10}(t)$.