6.2: Solving with the Laplace Transform

Key observation:
Using integration parts (with $u = e^{-st}$ and $dv = f'(t)dt$) yields: $\int e^{-st} f'(t)\, dt = f(t)e^{-st} + s \int e^{-st} f(t)\, dt$. Thus,

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt = -f(0) + s \int_0^\infty e^{-st} f(t)\, dt = -f(0) + s\mathcal{L}\{f(t)\}$$

Using the same idea, we can get formula for the higher derivatives. The results are:

1. $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$.
2. $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$.
3. $\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0)$.
4. $\mathcal{L}\{f^{(4)}(t)\} = s^4\mathcal{L}\{f(t)\} - s^3 f(0) - s^2 f'(0) - sf''(0) - f'''(0)$.

Using these facts, and the linearity discussed in 6.2, we can now take the Laplace transform of both sides of a differential equation as shown in the following examples:

1. Take the Laplace transform of both sides of $y'' + 4y = 0$ and simplify.

   Solution: Taking the Laplace transform of both sides yields: $\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = 0$.
   Using formula the formula from above yields: $s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4\mathcal{L}\{y\} = 0$.
   Simplifying gives $(s^2 + 4)\mathcal{L}\{y\} = sy(0) + y'(0)$.
   Thus, the Laplace transform of the solution will satisfy: $\mathcal{L}\{y\} = \frac{sy(0)+y'(0)}{s^2+4}$.

2. Take the Laplace transform of both sides of $y'' + 2y' + y = 4t$ and simplify.

   Solution: Taking the Laplace transform of both sides yields: $\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 4\mathcal{L}\{t\}$.
   Replacing using formulas from above and the formula for $\mathcal{L}\{t\} = \frac{1}{s^2}$ calculated in the previous section, we get $s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2(s\mathcal{L}\{y\} - y(0)) + \mathcal{L}\{y\} = \frac{4}{s^2}$.
   Simplifying gives $(s^2 + 2s + 1)\mathcal{L}\{y\} = (s + 2)y(0) + y'(0) + \frac{4}{s^2}$.
   Thus the Laplace transform of the solution will satisfy: $\mathcal{L}\{y\} = \frac{(s+2)y(0)+y'(0)}{s^2+2s+1} + \frac{4}{s^2(s^2+2s+1)}$.

Laplace Transform Solving Method
Given a constant coefficient linear differential equation:

1. Take the Laplace transform of both sides and solve for $\mathcal{L}\{y\}$. (Like in the examples above).
2. Use partial fractions to expand out your expressions.
3. Use the inverse Laplace transform table to look up the solution that corresponds to your expanded expressions.