3.6: Non-homogeneous Constant Coefficient Second Order (Variation of Parameters)

This is an alternative method to undetermined coefficients. The advantage of this method is that it is general (it applies to situations where $g(t)$ involves functions other than polynomials, exponentials, sines, and cosines). And this method also works for situations that are NOT constant coefficient. First, let me give you the 'shortcut' final answer, then the process will be explained on the following pages.

Variation of Parameter Theorem:
If $y_1(t)$ and $y_2(t)$ are independent solutions to the **homogeneous** equation $y'' + p(t)y' + g(t)y = 0$, then a particular solution to the **nonhomogeneous** equation $y'' + p(t)y' + q(t)y = g(t)$ can be written in the form:

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t), \quad \text{where} \quad u_1(t) = -\int \frac{g(t)y_2(t)}{W(y_1, y_2)}\, dt \quad \text{and} \quad u_2(t) = \int \frac{g(t)y_1(t)}{W(y_1, y_2)}\, dt,$$

where W is the Wronskian determinant $W = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t)$.
Furthermore, you can take $C = 0$ for both constants of integration.

Examples:

1. Give the general solution to $y'' + 10y' + 21y = 5e^{2t}$.
   (This is also the first example in my undetermined coefficients review, you can go back and compare the two approaches to see which one you prefer)

   Solution:

   (a) *Solve Homogeneous*:
   The equation $r^2 + 10r + 21 = (r + 3)(r + 7) = 0$ has the roots $r_1 = -3$ and $r_2 = -7$.
   So $y_1(t) = e^{-3t}$ and $y_2(t) = e^{-7t}$.

   (b) *Compute the Wronskian*:
   $W = \begin{vmatrix} e^{-3t} & e^{-7t} \\ -3e^{-3t} & -7e^{-7t} \end{vmatrix} = -7e^{-10t} + 3e^{-10t} = -4e^{-10t}$

   (c) *Use Variation of Parameters Theorem*:
   A particular solution can be found in the form $Y(t) = u_1(t)e^{-3t} + u_2(t)e^{-7t}$, where
   $u_1(t) = -\int \frac{5e^{2t}e^{-7t}}{-4e^{-10t}}\, dt = \int \frac{5}{4}e^{5t}\, dt = \frac{1}{4}e^{5t} + C_1.$
   $u_2(t) = \int \frac{5e^{2t}e^{-3t}}{-4e^{-10t}}\, dt = \int -\frac{5}{4}e^{9t}\, dt = -\frac{5}{36}e^{9t} + C_2.$
   Taking $C_1 = C_2 = 0$, we find a particular solution:
   $Y(t) = \frac{1}{4}e^{5t}e^{-3t} - \frac{5}{36}e^{9t}e^{-7t} = (\frac{1}{4} - \frac{5}{36})e^{2t} = \frac{9-5}{36}e^{2t} = \frac{1}{9}e^{2t}.$

   (d) *General Solution*:
   $y(t) = c_1e^{-3t} + c_2e^{-7t} + \frac{1}{9}e^{2t}.$

2. Give the general solution to $y'' - 2y' + y = 6t$.
(This is the second example in my undetermined coefficients review, again can go back and compare the two approaches to see which you prefer)

Solution:

(a) *Solve Homogeneous*:
The equation $r^2 - 2r + 1 = (r-1)^2 = 0$ has the one root $r = 1$.
So $y_1(t) = e^t$ and $y_2(t) = te^t$.

(b) *Compute the Wronskian*:
$W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t} = e^{2t}$

(c) *Use Variation of Parameters Theorem*:
A particular solution can be found in the form $Y(t) = u_1(t)e^t + u_2(t)te^t$, where
$u_1(t) = -\int \frac{6t^2e^t}{e^{2t}}\,dt = \int -6t^2e^{-t}\,dt = 6t^2e^{-t} + 12te^{-t} + 12e^{-t} + C_1$.
$u_2(t) = \int \frac{6te^t}{e^{2t}}\,dt = \int 6te^{-t}\,dt = -6te^{-t} - 6e^{-t} + C_2$.
Taking $C_1 = C_2 = 0$, we find a particular solution:
$Y(t) = (6t^2e^{-t} + 6te^{-t} + 12e^{-t})e^t + (-6te^{-t} - 6e^{-t})te^t = 6t^2 + 12t + 12 - 6t^2 - 6t = 6t + 12$.

(d) *General Solution*:
$y(t) = c_1e^t + c_2te^t + 6t + 12$.

3. Give the general solution to $y'' + y = \csc(t)$.
(This is NOT an example we did with undetermined coefficients. In fact we gave no way to do this in our discussion in section 3.5).

Solution:

(a) *Solve Homogeneous*:
The equation $r^2 + 1 = 0$ has the two roots $r = \pm i$.
So $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$.

(b) *Compute the Wronskian*:
$W = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = \cos^2(t) + \sin^2(t) = 1$

(c) *Use Variation of Parameters Theorem*:
A particular solution can be found in the form $Y(t) = u_1(t)\cos(t) + u_2(t)\sin(t)$, where
$u_1(t) = -\int \csc(t)\sin(t)\,dt = \int -1\,dt = -t + C_1$.
$u_2(t) = \int \csc(t)\cos(t)\,dt = \int \frac{\cos(t)}{\sin(t)}\,dt = \ln(\sin(t)) + C_2$.
Taking $C_1 = C_2 = 0$, we find a particular solution:
$Y(t) = -t\cos(t) + \ln(\sin(t))\sin(t)$.

(d) *General Solution*:
$y(t) = c_1\cos(t) + c_2\sin(t) - t\cos(t) + \ln(\sin(t))\sin(t)$.

4. You are told that $y_1(t) = t^3$ and $y_2(t) = t^4$ are independent solutions to the **homogeneous** equation $t^2y'' - 6ty' + 12y = 0$. (see section 3.4 for a discussion of this equation).
Use the variation of parameters theorem to find the general solution of the **nonhomogeneous** equation $t^2y'' - 6ty' + 12y = 7t$.

Notes before the solution: This is NOT constant coefficient, so our only method to try is variation of parameters. Also note that we need to write the equation in the form $y'' + p(t)y' + q(t)y = g(t)$. Dividing by t^2 gives $y'' - \frac{6}{t}y' + \frac{12}{t^2}y = \frac{7}{t}$. Thus, $g(t) = 7t^{-1}$.

Solution:

(a) *Compute the Wronskian*:
$$W = \begin{vmatrix} t^3 & t^4 \\ 3t^2 & 4t^3 \end{vmatrix} = 4t^6 - 3t^6 = t^6$$

(b) *Use Variation of Parameters Theorem*:
A particular solution can be found in the form $Y(t) = u_1(t)t^3 + u_2(t)t^4$, where
$$u_1(t) = -\int \frac{7t^{-1} \cdot t^4}{t^6}\, dt = \int -\frac{7}{t^3}\, dt = \frac{7}{2t^2} + C_1.$$
$$u_2(t) = \int \frac{7t^{-1} \cdot t^3}{t^6}\, dt = \int \frac{7}{t^4}\, dt = -\frac{7}{3t^3} + C_2.$$
Taking $C_1 = C_2 = 0$, we find a particular solution:
$Y(t) = \frac{7}{2t^2} \cdot t^3 - \frac{7}{3t^3} \cdot t^4 = \frac{7}{2}t - \frac{7}{3}t = \frac{7}{6}t.$

(c) *General Solution*:
$y(t) = c_1t^3 + c_2t^4 + \frac{7}{6}t.$

Explanation of Variation of Parameters: The goal is to solve $y'' + p(t)y' + q(t)y = g(t)$.

Step 1: For constant coefficient, find the general solution of the homogeneous equation $ay''+by'+cy = 0$. (Write and solve the characteristic equation, then use methods from 3.1, 3.3, and 3.4).
At this point, you'll have two independent solutions to the homogeneous equation: $y_1(t)$ and $y_2(t)$.
For a situation that is not constant coefficient, you will have to be given $y_1(t)$ and $y_2(t)$.

Step 2: Write $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$. We will search for functions $u_1(t)$ and $u_2(t)$ that make $Y(t)$ a solution.
Compute $Y'(t)$ and $Y''(t)$ AND make the simplifying assumption that $u_1'y_1 + u_2'y_2 = 0$.
In general: $Y' = u_1y_1' + u_2y_2' + u_1'y_1 + u_2'y_2$, and with the assumption, we get $Y' = u_1y_1' + u_2y_2'$.
Also $Y'' = u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2'$.

Step 3: Substitute into the equation.
In general: Substituting gives $(u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2') + p(u_1y_1' + u_2y_2') + q(u_1y_1 + u_2y_2) = g(t)$.
Since $y_1'' + py_1' + qy_1 = 0$ and $y_2'' + py_2' + qy_2 = 0$, this simplifies to $u_1'y_1' + u_2'y_2' = g(t)$.
At this point we are trying to solve for u_1 and u_2 and we know that:

$$\begin{aligned} u_1'(t)y_1(t) + u_2'(t)y_2(t) &= 0 \\ u_1'(t)y_1'(t) + u_2'(t)y_2'(t) &= g(t) \end{aligned}$$

This is a linear two-by-two system of equations, we can solve for u_1' and u_2' by combining and solving.

Step 4: Combine and solve to get u_1' and u_2'.
In general: Cramer's rule gives a compact way to write this (see my two-by-two system review for a discussion of this).
Using Cramer's rule we can write $u_1'(t) = \dfrac{-g(t)y_2(t)}{W(y_1, y_2)}$ and $u_2'(t) = \dfrac{g(t)y_1(t)}{W(y_1, y_2)}$, where $W(y_1, y_2)$ is the Wronskian.

Step 5: Integrate to get $u_1(t)$ and $u_2(t)$. (Since we are looking for *any* particular solution, let's use $C = 0$ for simplicity).
In general: $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = y_1(t)\int \frac{-g(t)y_2(t)}{W(y_1,y_2)}dt + y_2(t)\int \frac{g(t)y_1(t)}{W(y_1,y_2)}dt$.

Step 6: Give your general solution and solve for initial conditions.
General solution: $y(t) = c_1u_1(t) + c_2u_2(t) + Y(t)$.

Here is an example of the full process (this is the same as the first example on the first page of this review, except now I show all the steps).

Example:
Give the general solution to $y'' + 10y' + 21y = 5e^{2t}$.

Solution:

1. *Solve Homogeneous*:
   The equation $r^2 + 10r + 21 = (r+3)(r+7) = 0$ has the roots $r_1 = -3$ and $r_2 = -7$.
   So $y_1(t) = e^{-3t}$ and $y_2(t) = e^{-7t}$

2. *Set Up Variation of Parameters*:
   $Y(t) = u_1(t)e^{-3t} + u_2(t)e^{-7y}$.
   $Y'(t) = -3u_1e^{-3t} - 7u_2e^{-7t} + u_1'e^{-3t} + u_2'e^{-7t}$ and we make the additional assumption that $u_1'e^{-3t} + u_2'e^{-7t} = 0$.
   So $Y'(t) = -3u_1e^{-3t} - 7u_2e^{-7t}$.
   And $Y''(t) = -3u_1'e^{-3t} - 7u_2'e^{-7t} + 9u_1e^{-3t} + 49u_2e^{-7t}$.

3. *Substitution*:
   $(-3u_1'e^{-3t} - 7u_2'e^{-7t} + 9u_1e^{-3t} + 49u_2e^{-7t}) + 10(-3u_1e^{-3t} - 7u_2e^{-7t}) + 21(u_1e^{-3t} + u_2e^{-7t} = 5e^{2t}$.
   which simplifies to $-3u_1'e^{-3t} - 7u_2'e^{-7t} = 5e^{2t}$. Thus, we have two conditions:
   $$\begin{aligned} u_1'(t)e^{-3t} + u_2'(t)e^{-7t} &= 0 \\ -3u_1'(t)e^{-3t} - 7u_2'(t)e^{-7t} &= 5e^{2t} \end{aligned}$$

4. *Solve the two-by-two system*:
   Note $W(y_1, y_2) = -7e^{-10t} + 3e^{-10t} = -4e^{-10t}$.
   Using Cramer's rule (or just solving by combining), you get $u_1' = \frac{-g(t)y_2(t)}{W(y_1,y_2)} = -\frac{5e^{-5t}}{-4e^{-10t}} = \frac{5}{4}e^{5t}$ and $u_2' = \frac{5e^{-t}}{-4e^{-10t}} = -\frac{5}{4}e^{9t}$.

5. *Integrating*:
   $u_1(t) = \int \frac{5}{4}e^{5t}\,dt = \frac{1}{4}e^{5t} + C_1$ and $u_2(t) = \int \frac{5}{4}e^{9t}\,dt = -\frac{5}{36}e^{9t} + C_2$.
   Thus, $Y(t) = u_1(t)e^{-3t} + u_2(t)e^{-7y} = \frac{1}{4}e^{2t} - \frac{5}{36}e^{2t}$

6. $y(t) = c_1e^{-3t} + c_2e^{-7t} + \frac{1}{9}e^{2t}$.