

2.5: Autonomous Differential Equations and Equilibrium Analysis

An **autonomous** first order ordinary differential equation is any equation of the form: $\frac{dy}{dt} = f(y)$.

Note: In my home dictionary, the word “autonomous” is defined as “existing or acting separately from other things or people”. In the context of differential equations, autonomous means that the derivative can be expressed without any explicitly reference to time, t .

Solving Autonomous Equations:

Since there is no time t , ALL autonomous equations are separable! In other words, separating gives $\frac{1}{f(y)}dy = dt$ and integrating gives

$$\int \frac{1}{f(y)}dy = t + C.$$

Of course, we would need to be able to compute this integral!

Stable, Unstable and Semi-stable Equilibrium Solutions:

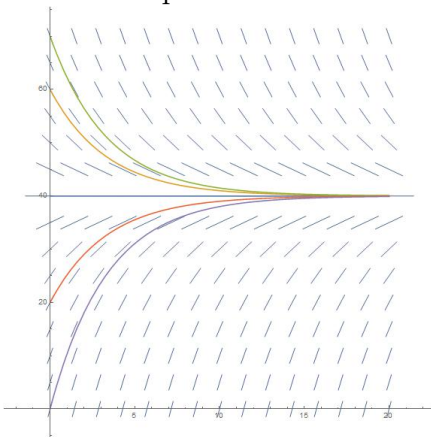
Recall that an **equilibrium solution** is any constant (horizontal) function $y(t) = c$ that is a solution to the differential equation. Notice that the derivative of a constant function is always 0, so we find equilibrium solutions by solving for y in the equation $\frac{dy}{dt} = f(t, y) = 0$.

For autonomous equations, using the general existence/uniqueness theorem of the last section, if $f(y)$ and $\frac{\partial f}{\partial y}$ have no discontinuities, then we are guaranteed that no other solution can intersect an equilibrium solution (if they did we wouldn't have uniqueness).

Thus, for situations when $f(y)$ and $\frac{\partial f}{\partial y}$ have no discontinuities, we can always classify the equilibrium solutions as follows:

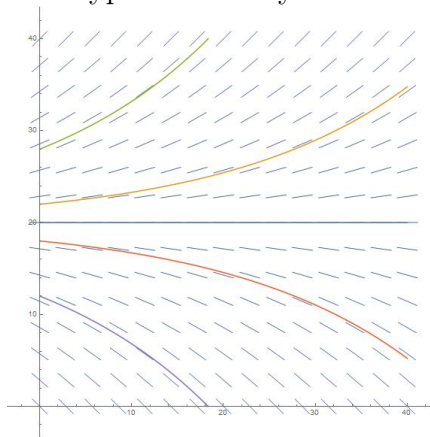
- **Stable:** The equilibrium solution $y(t) = c$ is **stable** if all solutions with initial conditions y_0 ‘near’ $y = c$ approach c as $t \rightarrow \infty$.
- **Unstable:** The equilibrium solution $y(t) = c$ is **unstable** if all solutions with initial conditions y_0 ‘near’ $y = c$ do NOT approach c as $t \rightarrow \infty$.
- **Semi-stable:** The equilibrium solution $y(t) = c$ is **semistable** if initial conditions y_0 on one side of c lead to solutions $y(t)$ that approach c as $t \rightarrow \infty$, while initial conditions y_0 on the other side of c do NOT approach c .

A basic example and illustration of each type of stability is shown below:



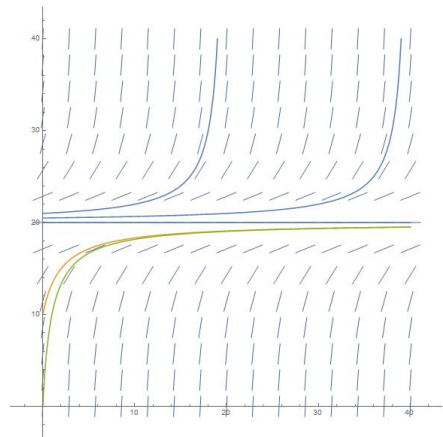
$$y(t) = 40 \text{ is STABLE}$$

$$\frac{dy}{dt} = -0.3(y - 40)$$



$$y(t) = 20 \text{ is UNSTABLE}$$

$$\frac{dy}{dt} = 0.3(y - 20)$$



$$y(t) = 20 \text{ is SEMISTABLE}$$

$$\frac{dy}{dt} = 0.05(y - 40)^2$$

Classifying Equilibrium Solutions:

Given $\frac{dy}{dt} = f(y)$. (and assuming f and $\frac{\partial f}{\partial y}$ are continuous)

1. Solve $f(y) = 0$ to get the equilibrium solutions.
2. Study $f(y)$ around the equilibrium values as follows (drawing $f(y)$ might help):
 - (a) Draw a vertical line (the *phase* line) and make tick marks at equilibrium values.
 - (b) Between tick marks determine if $f(y)$ is positive or negative.
 - (c) If $f(y)$ is **positive**, then $\frac{dy}{dt}$ will be positive so any solution in this region will be **increasing**.
 - (d) If $f(y)$ is **negative**, then $\frac{dy}{dt}$ will be negative so any solution in this region will be **decreasing**.
3. Classify:
 - (a) Increasing below and decreasing above \implies stable.
 - (b) Decreasing below and increasing above \implies unstable.
 - (c) Decreasing below and above OR increasing below and above \implies semistable.

Examples:

1. Find and classify the equilibrium points of $\frac{dy}{dt} = (1 - y)(3 - y)$.
 - (a) $y(t) = 1$ and $y(t) = 3$ are the equilibrium solutions.
 - (b) For $y > 3$: $\frac{dy}{dt}$ is positive, so $y(t)$ is increasing.
For $1 < y < 3$: $\frac{dy}{dt}$ is negative, so $y(t)$ is decreasing.
For $y < 1$: $\frac{dy}{dt}$ is positive, so $y(t)$ is increasing.
 - (c) Thus, $y(t) = 1$ is stable. And $y(t) = 3$ is unstable.
2. Find and classify the equilibrium points of $\frac{dy}{dt} = -(y - 10)^2(y - 4)$.
 - (a) $y(t) = 4$ and $y(t) = 10$ are the equilibrium solutions.
 - (b) For $y > 10$: $\frac{dy}{dt}$ is negative, so $y(t)$ is decreasing.
For $4 < y < 10$: $\frac{dy}{dt}$ is negative, so $y(t)$ is decreasing.
For $y < 4$: $\frac{dy}{dt}$ is positive, so $y(t)$ is increasing.
 - (c) Thus, $y(t) = 4$ is stable. And $y(t) = 10$ is semistable.
3. Find and classify the equilibrium points of $\frac{dy}{dt} = (y^3 - 8)(e^y - 1)$.
 - (a) $y(t) = 0$, $y(t) = 2$ are the equilibrium solutions.
 - (b) For $y > 2$: $\frac{dy}{dt}$ is positive, so $y(t)$ is increasing.
For $0 < y < 2$: $\frac{dy}{dt}$ is negative, so $y(t)$ is decreasing.
For $y < 0$: $\frac{dy}{dt}$ is positive, so $y(t)$ is increasing.
 - (c) Thus, $y(t) = 0$ is stable. And $y(t) = 2$ is unstable.

Population Dynamics:

Typically, the rate of change of a population is only dependent on the current population size in some way. Thus, the differential equation for a population is typically time-independent (so it is autonomous). The study of populations is a big application of differential equations that we have been waiting to discuss until now.

1. Let $y = y(t)$ = the size of the population at time t .
This could be any organism (people, people that have disease, people that know a rumor, bugs, bacteria, fish, other animals, plants, ...).

2. Thus, $\frac{dy}{dt}$ = rate of change of population with respect to time.

For example, if y is in bacteria and t is in minutes, then the units of $\frac{dy}{dt}$ will be bacteria/minute. So $\frac{dy}{dt}$ can, roughly, be thought of as the number of bacteria that will be added to the population over the next minute (I say 'roughly' because it actually is the instantaneous rate which is not exactly the same as the average rate over the next minute, but they would be very, very close).

We considered three models in lecture:

- Unrestricted (Natural) Growth: $\frac{dy}{dt} = ry$
 - Plenty of food and space. Assumes: "The rate of change of the population proportional to the population size."
 - r = relative growth rate (decimal version of the percentage growth per time). So ry is the approximate number of people added to the population each year.
 - The solution is exponential.
 - There is only one equilibrium at $y(t) = 0$ and, for positive r , it is unstable.
- Logistic Growth: $\frac{dy}{dt} = r \left(1 - \frac{y}{K}\right) y$, with $0 < K$ and $r > 0$.
 - Initially like unrestricted growth, but there is a carrying capacity K that the population size can't exceed.
 - For positive r , there are two equilibrium at $y(t) = 0$ which is unstable and $y(t) = K$ which is stable.
- Logistic Growth with a Threshold: $\frac{dy}{dt} = -r \left(1 - \frac{y}{K}\right) \left(1 - \frac{y}{T}\right) y$ with $0 < T < K$, $r > 0$
 - If the population size is below a certain threshold size, T , then the population will decrease to zero. If the population is above this threshold, then the population behaves like logistic growth.
 - For positive r , there are three equilibrium at $y(t) = 0$ which is stable, $y(t) = T$ which is unstable, and $y(t) = K$ which is stable.