2.1: Integrating Factors

Some Observations and Motivation:

1. The first observation is the product rule: \( \frac{d}{dt} \left( f(t)y \right) = f(t) \frac{dy}{dt} + f'(t)y. \)

Here are a couple of quick derivative examples (we are assuming \( y \) is a function of \( t \)):

\[
\frac{d}{dt} (t^3 y) = t^3 \frac{dy}{dt} + 3t^2 y \quad \text{and} \quad \frac{d}{dt} (e^{4t} y) = e^{4t} \frac{dy}{dt} + 4e^{4t} y.
\]

Thus, \( f(t) \frac{dy}{dt} + f'(t)y = g(t) \) can be rewritten as \( \frac{d}{dt} (f(t)y) = g(t). \)

2. The second observation (using the chain rule with \( e^{F(x)} \)):

\[
\frac{d}{dt} (e^{F(t)} y) = e^{F(t)} \frac{dy}{dt} + F'(t)e^{F(t)} y.
\]

Integrating Factor Method:

If we start with \( \frac{dy}{dt} + p(t)y = g(t) \) AND if we can find an antiderivative of \( p(t) \), then we can use the following process:

1. First rewrite the differential equation in the form: \( \frac{dy}{dt} + p(t)y = g(t) \)

2. Find any antiderivative of \( p(t) \) and write \( \mu(t) = e^{\int p(t) \, dt} \)

3. Multiply the entire equation by \( \mu(t) \) and use the facts from above, so

\[
\frac{dy}{dt} + p(t)y = g(t) \quad \text{becomes} \quad \mu(t) \frac{dy}{dt} + p(t)\mu(t)y = g(t)\mu(t) \quad \text{which becomes} \quad \frac{d}{dt} (\mu(t)y) = g(t)\mu(t)
\]

4. Integrate with respect to \( t \) and you are done! (Of course, as always, also simplify, use initial conditions and check your work)

NOTES:

1. This is a method for first order linear differential equations. Meaning you can only have \( y \) to the first power, and nothing else in terms of \( y \).

2. Using the substitution idea that I introduced in the previous section, you can sometimes turn a nonlinear problem into a linear problem. Here are two examples:

   • Using \( u = e^y \) on the equation \( e^y \frac{dy}{dx} - xe^y = 2x \) yields the linear equation \( \frac{du}{dx} - xu = 2x \).
   
   • Using \( u = \ln(y) \) on the equation \( \frac{1}{y} \frac{dy}{dx} - \frac{\ln(y)}{x} = x \) yields the linear equation \( \frac{du}{dx} - \frac{u}{x} = x \).

3. A small note about the form of some answers from the textbook:

   When we are unable to integrate a function in an elementary way, you will sometimes see an answer written in the following form \( \int f(x) \, dx = \int_{x_0}^{x} f(u) \, du + C \), where \( x_0 \) is the \( x \)-value of some initial condition.

   There is nothing scary happening here, let me give you an example to ease your mind.

   Consider \( \int x^2 \, dx \) and \( \int_{0}^{x} u^2 \, du + C \). Let me compute both:

   \[
   \int x^2 \, dx = \frac{1}{3} x^3 + C \quad \text{and} \quad \int_{0}^{x} u^2 \, du + C = \frac{1}{3} u^3 \bigg|_{0}^{x} + C = \frac{1}{3} x^3 + C.
   \]

   Notice they are the same. This gives a way to explicitly include your initial condition ‘+C’ in writing down your final answer even if you can’t integrate.
Integrating Factor Examples:

1. Find the explicit solution to \(4\frac{dy}{dt} - 8y = 4e^{5t}\) with \(y(0) = \frac{2}{3}\).

   **Solution:**
   
   (a) Rewrite: \(\frac{dy}{dt} - 2y = e^{5t}\), so \(p(t) = -2, g(t) = e^{5t}\).
   
   (b) Integrating Factor: \(\int p(t)\,dt = \int -2\,dt = -2t + C, \mu(t) = e^{-2t}\).
   
   (c) Multiply: \(\frac{dy}{dt} + 2y = e^{5t}\) becomes \(e^{-2t}\frac{dy}{dt} + 2e^{-2t}y = e^{3t}\) which becomes \(\frac{d}{dt} (e^{-2t}y) = e^{3t}\).
   
   (d) Integrate: \(e^{-2t}y = \int e^{3t}\,dt = \frac{1}{3}e^{3t} + C, \) so \(y = \frac{1}{3}e^{5t} + Ce^{2t}\).

   Using the initial condition gives, \(\frac{2}{3} = \frac{1}{3} + C, \) so \(C = \frac{1}{3}\).

   For a final answer of \(y = \frac{1}{3}e^{5t} + \frac{1}{3}e^{2t}\).

2. Find the explicit solution to \(t\frac{dy}{dt} + 2y = \cos(t)\) with \(y(\pi) = 1\).

   **Solution:**
   
   (a) Rewrite: \(\frac{dy}{dt} + \frac{2}{t}y = \frac{\cos(t)}{t}\), so \(p(t) = \frac{2}{t}, g(t) = \frac{\cos(t)}{t}\).
   
   (b) Integrating Factor: \(\int p(t)\,dt = \int \frac{2}{t}\,dt = 2 \ln |t| + C = \ln(t^2) + C, \) so \(\mu(t) = e^{\ln(t^2)} = t^2\).
   
   (c) Multiply: \(\frac{dy}{dt} + \frac{2}{t}y = \frac{\cos(t)}{t}\) becomes \(t^2\frac{dy}{dt} + 2ty = t \cos(t)\) which becomes \(\frac{d}{dt} (t^2y) = t \cos(t)\).
   
   (d) Integrate: \(t^2y = \int t \cos(t)\,dt = t \sin(t) + \cos(t) + C\) (using by parts), so \(y = \frac{\sin(t)}{t} + \frac{\cos(t)}{t^2} + \frac{C}{t^2}\).

   Using the initial condition gives, \(1 = 0 - \frac{1}{t} + \frac{C}{t^2} + C, \) so \(C = t^2 + 1\).

   For a final answer of \(y = \frac{\sin(t)}{t} + \frac{\cos(t)}{t^2} + \frac{(t^2+1)}{t^2}\).

3. Find the explicit solution to \(\cos(y)\frac{dy}{dt} - \frac{\sin(y)}{t} = t\) with \(y(2) = 0\). (Hint: Start with \(u = \sin(y)\))

   **Solution:**
   
   Using \(u = \sin(y)\) we get \(\frac{du}{dt} = \cos(y)\frac{dy}{dt}\), so the differential equation can be rewritten at \(\frac{du}{dt} - \frac{1}{t}u = t\).

   Now we will solve this:
   
   (a) Rewrite: \(p(t) = -\frac{1}{t}, g(t) = t\).
   
   (b) Integrating Factor: \(\int p(t)\,dt = \int -\frac{1}{t}\,dt = -\ln(t) + C = \ln(\frac{1}{t}) + C, \) so \(\mu(t) = e^{\ln(1/t)} = \frac{1}{t}\).
   
   (c) Multiply: \(\frac{du}{dt} - \frac{1}{t}u = t\) becomes \(\frac{1}{t^2}\frac{du}{dt} - \frac{1}{t^2}u = 1\) which becomes \(\frac{d}{dt} \left(\frac{1}{t}u\right) = 1\).
   
   (d) Integrate: \(\frac{1}{t}u = \int 1\,dt = t + C, \) so \(u = t^2 + Ct\).

   Going back to \(y\) gives \(\sin(y) = t^2 + Ct\).

   Using the initial condition gives, \(\sin(0) = 2^2 + 2C, \) so \(C = -2\).

   For an answer of \(\sin(y) = t^2 - 2t, \) or \(y = \sin^{-1}(t^2 - 2t)\).