2.1: Integrating Factors

Some Observations and Motivation:

1. The first observation is the product rule: $\frac{d}{dt}(f(t)\,y) = f(t)\frac{dy}{dt} + f'(t)y$.
   Here are a couple of quick derivative examples (we are assuming y is a function of t):
   $$\frac{d}{dt}\left(t^3\,y\right) = t^3\frac{dy}{dt} + 3t^2 y \quad \text{and} \quad \frac{d}{dt}\left(e^{4t}\,y\right) = e^{4t}\frac{dy}{dt} + 4e^{4t}y.$$
   $$\textbf{Thus,}\quad f(t)\frac{dy}{dt} + f'(t)y = g(t) \quad \textbf{can rewritten as} \quad \frac{d}{dt}(f(t)y) = g(t).$$

2. The second observation (using the chain rule with $e^{F(x)}$): $\frac{d}{dt}\left(e^{F(t)}\,y\right) = e^{F(t)}\frac{dy}{dt} + F'(t)e^{F(t)}y$.

Integrating Factor Method:
If we start with $\frac{dy}{dt} + p(t)y = g(t)$ AND if we can find an antiderivative of $p(t)$, then we can use the following process:

1. First rewrite the differential equation in the form: $\frac{dy}{dt} + p(t)y = g(t)$
2. Find **any** antiderivative of $p(t)$ and write $\mu(t) = e^{\int p(t)\,dt}$
3. Multiply the entire equation by $\mu(t)$ and use the facts from above, so
   $$\frac{dy}{dt} + p(t)y = g(t) \quad \textbf{becomes} \quad \mu(t)\frac{dy}{dt} + p(t)\mu(t)y = g(t)\mu(t) \quad \textbf{which becomes} \quad \frac{d}{dt}(\mu(t)y) = g(t)\mu(t)$$
4. Integrate with respect to t and you are done! (Of course, as always, also simplify, use initial conditions and check your work)

NOTES:

1. This is a method for first order **linear** differential equations. Meaning you can only have y to the first power, and nothing else in terms of y.
2. Using the substitution idea that I introduced in the previous section, you can sometimes turn a nonlinear problem into a linear problem. Here are two examples:
   - Using $u = e^y$ on the equation $e^y\frac{dy}{dx} - xe^y = 2x$ yields the linear equation $\frac{du}{dx} - xu = 2x$.
   - Using $u = \ln(y)$ on the equation $\frac{1}{y}\frac{dy}{dx} - \frac{\ln(y)}{x} = x$ yields the linear equation $\frac{du}{dx} - \frac{u}{x} = x$.
3. A small note about the form of some answers from the textbook:
   When we are unable to integrate a function in an elementary way, you will sometimes see an answer written in the following form $\int f(x)dx = \int_{x_0}^{x} f(u)\,du + C$, where x_0 is the x-value of some initial condition.
   There is nothing scary happening here, let me give you can example to ease your mind.
   Consider $\int x^2\,dx$ and $\int_0^x u^2\,du + C$. Let me compute both:
   $$\int x^2\,dx = \frac{1}{3}x^3 + C \quad \text{and} \quad \int_0^x u^2\,du + C = \frac{1}{3}u^3\Big|_0^x + C = \frac{1}{3}x^3 + C.$$
   Notice they are the same. This gives a way to explicitly include your initial condition '+C' in writing down your final answer even if you can't integrate.

Integrating Factor Examples:

1. Find the explicit solution to $4\dfrac{dy}{dt} - 8y = 4e^{5t}$ with $y(0) = \frac{2}{3}$.

   Solution:

   (a) Rewrite: $\dfrac{dy}{dt} - 2y = e^{5t}$, so $p(t) = -2$, $g(t) = e^{5t}$.

   (b) Integrating Factor: $\int p(t)\,dt = \int -2\,dt = -2t + C$, so $\mu(t) = e^{-2t}$.

   (c) Multiply: $\frac{dy}{dt} + 2y = e^{5t}$ becomes $e^{-2t}\frac{dy}{dt} + 2e^{-2t}y = e^{3t}$ which becomes $\frac{d}{dt}\left(e^{-2t}y\right) = e^{3t}$.

   (d) Integrate: $e^{-2t}y = \int e^{3t}dt = \frac{1}{3}e^{3t} + C$, so $y = \frac{1}{3}e^{5t} + Ce^{2t}$.
   Using the initial condition gives, $\frac{2}{3} = \frac{1}{3} + C$, so $C = \frac{1}{3}$.
   For a final answer of $y = \frac{1}{3}e^{5t} + \frac{1}{3}e^{2t}$.

2. Find the explicit solution to $t\dfrac{dy}{dt} + 2y = \cos(t)$ with $y(\pi) = 1$.

   Solution:

   (a) Rewrite: $\dfrac{dy}{dt} + \dfrac{2}{t}y = \dfrac{\cos(t)}{t}$, so $p(t) = \frac{2}{t}$, $g(t) = \frac{\cos(t)}{t}$.

   (b) Integrating Factor: $\int p(t)\,dt = \int \frac{2}{t}\,dt = 2\ln|t| + C = \ln(t^2) + C$, so $\mu(t) = e^{\ln(t^2)} = t^2$.

   (c) Multiply: $\frac{dy}{dt} + \frac{2}{t}y = \frac{\cos(t)}{t}$ becomes $t^2\frac{dy}{dt} + 2ty = t\cos(t)$ which becomes $\frac{d}{dt}\left(t^2y\right) = t\cos(t)$.

   (d) Integrate: $t^2y = \int t\cos(t)dt = t\sin(t) + \cos(t) + C$ (using by parts), so $y = \frac{\sin(t)}{t} + \frac{\cos(t)}{t^2} + \frac{C}{t^2}$.
   Using the initial condition gives, $1 = 0 - \frac{1}{\pi^2} + \frac{C}{\pi^2} + C$, so $C = \pi^2 + 1$.
   For a final answer of $y = \frac{\sin(t)}{t} + \frac{\cos(t)}{t^2} + \frac{(\pi^2+1)}{t^2}$.

3. Find the explicit solution to $\cos(y)\dfrac{dy}{dt} - \dfrac{\sin(y)}{t} = t$ with $y(2) = 0$. (Hint: Start with $u = \sin(y)$)

   Solution:
   Using $u = \sin(y)$ we get $\frac{du}{dt} = \cos(y)\frac{dy}{dt}$, so the differential equation can be rewritten at $\frac{du}{dt} - \frac{1}{t}u = t$.
   Now we will solve this:

   (a) Rewrite: $p(t) = -\frac{1}{t}$, $g(t) = t$.

   (b) Integrating Factor: $\int p(t)\,dt = \int -\frac{1}{t}\,dt = -\ln(t) + C = \ln\left(\frac{1}{t}\right) + C$, so $\mu(t) = e^{\ln(1/t)} = \frac{1}{t}$.

   (c) Multiply: $\frac{du}{dt} - \frac{1}{t}u = t$ becomes $\frac{1}{t}\frac{du}{dt} - \frac{1}{t^2}u = 1$ which becomes $\frac{d}{dt}\left(\frac{1}{t}u\right) = 1$.

   (d) Integrate: $\frac{1}{t}u = \int 1\,dt = t + C$, so $u = t^2 + Ct$.
   Going back to y gives $\sin(y) = t^2 + Ct$.
   Using the initial condition gives, $\sin(0) = 2^2 + 2C$, so $C = -2$.
   For an answer of $\sin(y) = t^2 - 2t$, or $y = \sin^{-1}(t^2 - 2t)$.