Laplace Transform Fact Sheet

General and Important Facts:

	General Result	Examples
Definition :	$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\, dt$	
Linearity :	$\mathcal{L}\{c_1 y_1(t) + c_2 y_2(t)\} = c_1\mathcal{L}\{y_1(t)\} + c_2\mathcal{L}\{y_2(t)\}$	$\mathcal{L}\{3e^{2t} - 5t^2\} = 3\mathcal{L}\{e^{2t}\} - 5\mathcal{L}\{t^2\}$
Linearity :	$\mathcal{L}^{-1}\{c_1 F(s) + c_2 G(s)\} = c_1\mathcal{L}^{-1}\{F(s)\} + c_2\mathcal{L}^{-1}\{G(s)\}$	$\mathcal{L}^{-1}\{\frac{2}{s+1} - \frac{4}{s}\} = 2\mathcal{L}^{-1}\{\frac{1}{s+1}\} - 4\mathcal{L}^{-1}\{\frac{1}{s}\}$
1st Deriv. :	$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - f(0)$	
2nd Deriv. :	$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sf(0) - f'(0)$	
Exponentials :	$\mathcal{L}\{e^{ct} f(t)\}(s) = \mathcal{L}\{f(t)\}(s-c)$	$\mathcal{L}\{e^{5t}\sin(2t)\}(s) = \mathcal{L}\{\sin(2t)\}(s-5)$
Polynomials :	$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\mathcal{L}\{f(t)\}(s)\right)$	$\mathcal{L}\{t\sin(5t)\} = -\frac{d}{ds}\left(\mathcal{L}\{\sin(5t)\}\right)$
Unit step :	$u_c(t) = \begin{cases} 0, & t < c; \\ 1, & t \geq c. \end{cases}$	$u_2(t) = \begin{cases} 0, & t < 2; \\ 1, & t \geq 2. \end{cases}$
Unit step :	$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$	$\mathcal{L}\{u_3(t) e^{t-3}\} = e^{-3s}\mathcal{L}\{e^t\}$

Elementary Laplace Transform Table: Here n is a positive integer.

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	*Examples/Notes* :
1	$\frac{1}{s}$	$\mathcal{L}\{6\} = 6\mathcal{L}\{1\} = \frac{6}{s}$
e^{at}	$\frac{1}{s-a}$	$\mathcal{L}\{4e^{5t}\} = \frac{4}{s-5}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$	$\mathcal{L}\{7\cos\left(\frac{1}{2}t\right)\} = \frac{7s}{s^2+1/4}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$	$\mathcal{L}\{5\sin(3t)\} = \frac{5\cdot 3}{s^2+9} = \frac{15}{s^2+9}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$\mathcal{L}\{6e^{2t}\cos(t)\} = \frac{6(s-2)}{(s-2)^2+1}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$\mathcal{L}\{2e^{t}\sin(5t)\} = \frac{2\cdot 5}{(s-1)^2+25} = \frac{10}{(s-1)^2+25}$
t^n	$\frac{n!}{s^{n+1}}$	$\mathcal{L}\{t\} = \frac{1}{s^2}$, $\mathcal{L}\{t^3\} = \frac{3!}{t^4}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$\mathcal{L}\{7t^2 e^{8t}\} = \frac{7\cdot 2}{(s-8)^2} = \frac{14}{(s-8)^2}$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$\mathcal{L}\{9u_2(t)\} = \frac{9e^{-2s}}{s}$
$u_c(t) f(t-c)$	$e^{-cs}F(s)$	$\mathcal{L}\{u_3(t)e^{t-3}\} = e^{-3s}\mathcal{L}\{e^t\} = \frac{e^{-3s}}{s-1}$
$(e^{bt} + e^{-bt})/2 = \cosh(bt)$	$\frac{s}{s^2-b^2}$	$\mathcal{L}\{\cosh(2t)\} = \frac{s}{s^2-4}$
$(e^{bt} - e^{-bt})/2 = \sinh(bt)$	$\frac{b}{s^2-b^2}$	$\mathcal{L}\{\sinh(3t)\} = \frac{3}{s^2-9}$
t^p	$\frac{\Gamma(p+1)}{s^{p+1}}$	$\Gamma(p+1) = \int_0^\infty t^p e^{-t}\, dt$ is the Gamma function
$\delta(t-c)$	e^{-cs}	$\delta(t-c)$ is the unit impulse function at $t = c$

Laplace Transform Method:
To solve $ay'' + by' + cy = g(t)$, where $g(t)$ can be any forcing function (we even discuss how it can have discontinuities).

1. **Take the Laplace transform of both sides**.
   Since the transform is linear, we get $a\mathcal{L}\{y''\} + b\mathcal{L}\{y'\} + c\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$.

2. **Use the rules for the 1st and 2nd derivative and solve for $\mathcal{L}\{y\}$**.
   Since $\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - f(0)$ and $\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sf(0) - f'(0)$,
   we get $(as^2 + bs + c)\mathcal{L}\{y\} - (as + b)f(0) - af'(0) = \mathcal{L}\{g(t)\}$.
   Also replace $\mathcal{L}\{g(t)\}$ by its Laplace transform. Now solve for $\mathcal{L}\{y\}$.

3. **Partial Fractions**:
   Break up the expression you found into partial fractions.

4. **Look in the table for the inverse Laplace transform**:
   Look up the answers in the table.

Elementary Inverse Laplace Transform Table: Most inverses are easy to get from the previous table, but sometimes you may have to rewrite the expression in order to make it look like the table. Below I have rewritten the rules from the previous page in terms of the inverse.

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$Examples/Notes:$
$\mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at}$	$\mathcal{L}^{-1}\{\frac{1}{s-5}\} = e^{5t}$
$\mathcal{L}^{-1}\{\frac{s}{s^2+b^2}\} = \cos(bt)$	$\mathcal{L}^{-1}\{\frac{s}{s^2+4}\} = \cos(2t)$
$\mathcal{L}^{-1}\{\frac{1}{s^2+b^2}\} = \frac{1}{b}\sin(bt)$	$\mathcal{L}^{-1}\{\frac{1}{s^2+9}\} = \frac{1}{3}\sin(3t)$
$\mathcal{L}^{-1}\{\frac{(s-a)}{(s-a)^2+b^2}\} = e^{at}\cos(bt)$	$\mathcal{L}^{-1}\{\frac{s-3}{(s-3)^2+16}\} = e^{3t}\cos(4t)$
$\mathcal{L}^{-1}\{\frac{1}{(s-a)^2+b^2}\} = \frac{1}{b}e^{at}\sin(bt)$	$\mathcal{L}^{-1}\{\frac{1}{(s+1)^2+25}\} = \frac{1}{5}e^{-t}\cos(5t)$
$\mathcal{L}^{-1}\{\frac{1}{s^n}\} = \frac{1}{(n-1)!}t^{n-1}$	$\mathcal{L}^{-1}\{\frac{1}{s^3}\} = \frac{1}{2}t^2$
$\mathcal{L}^{-1}\{\frac{1}{(s-a)^n}\} = \frac{1}{(n-1)!}t^{n-1}e^{at}$	$\mathcal{L}^{-1}\{\frac{1}{(s+2)^4}\} = \frac{1}{3!}t^3e^{-2t}$
$\mathcal{L}^{-1}\{\frac{e^{-cs}}{s}\} = u_c(t)$	$\mathcal{L}^{-1}\{\frac{e^{-7s}}{s}\} = u_7(t)$
$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)\mathcal{L}^{-1}\{F(s)\}(t-c)$	$\mathcal{L}^{-1}\{\frac{e^{-7s}}{s-4}\} = u_7(t)\mathcal{L}^{-1}\{\frac{1}{s-4}\}(t-7) = u_7(t)e^{4(t-c)}$

Examples: Try these on your own before you look at the solutions (solutions on the next page).

1. Solve $y'' + 3y' - 4y = 0$ with $y(0) = 0$ and $y'(0) = 6$, using the Laplace transform.
2. Solve $y'' + 2y + y = 0$ with $y(0) = 3$ and $y'(0) = 1$, using the Laplace transform.
3. Solve $y'' - y = e^{2t}$ with $y(0) = 0$ and $y'(0) = 1$, using the Laplace transform.
4. Solve $y'' + y = u_5(t)$ with $y(0) = 0$ and $y'(0) = 3$, using the Laplace transform.

Solutions to examples:

1. Solve $y'' + 3y' - 4y = 0$ with $y(0) = 0$ and $y'(0) = 6$, using the Laplace transform.

   (a) *Laplace Transform*: $\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} - 4\mathcal{L}\{y\} = \mathcal{L}\{0\}$.

   (b) *Use Rules and Solve*: $s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 3s\mathcal{L}\{y\} - 3y(0) - 4\mathcal{L}\{y\} = 0$, which becomes: $(s^2 + 3s - 4)\mathcal{L}\{y\} - 6 = 0$. Solving for $\mathcal{L}\{y\}$ gives: $\mathcal{L}\{y\} = \frac{6}{s^2+3s-4}$.

   (c) *Partial Fractions*: $\frac{6}{s^2+3s-4} = \frac{6}{(s+4)(s-1)} = \frac{A}{s+4} + \frac{B}{s-1}$ and you find $A = -\frac{6}{5}$, $B = \frac{6}{5}$.

   (d) *Inverse Laplace transform*: The solution is: $y(t) = \mathcal{L}^{-1}\{\frac{-6/5}{s+4} + \frac{6/5}{s-1}\} = -\frac{6}{5}e^{-4t} + \frac{6}{5}e^{t}$.

2. Solve $y'' + 2y + y = 0$ with $y(0) = 3$ and $y'(0) = 1$, using the Laplace transform.

   (a) *Laplace Transform*: $\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{0\}$.

   (b) *Use Rules and Solve*: $s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2s\mathcal{L}\{y\} - 2y(0) + \mathcal{L}\{y\} = 0$, which becomes: $(s^2 + 2s + 1)\mathcal{L}\{y\} - (7 + 3s) = 0$. Solving for $\mathcal{L}\{y\}$ gives: $\mathcal{L}\{y\} = \frac{3s+7}{s^2+2s+1}$.

   (c) *Partial Fractions*: $\frac{3s+7}{s^2+2s+1} = \frac{3s+7}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$ and you find $A = 3$, $B = -4$.

   (d) *Inverse Laplace transform*: The solution is: $y(t) = \mathcal{L}^{-1}\{\frac{3}{s+1} + \frac{-4}{(s+1)^2}\} = 3e^{-t} - 4te^{-t}$.

3. Solve $y'' - y = e^{2t}$ with $y(0) = 0$ and $y'(0) = 1$, using the Laplace transform.

   (a) *Laplace Transform*: $\mathcal{L}\{y''\} - \mathcal{L}\{y\} = \mathcal{L}\{e^{2t}\}$.

   (b) *Use Rules and Solve*: $s^2\mathcal{L}\{y\} - sy(0) - y'(0) - \mathcal{L}\{y\} = \frac{1}{s-2}$, which becomes: $(s^2 - 1)\mathcal{L}\{y\} - 1 = \frac{1}{s-2}$. Solving for $\mathcal{L}\{y\}$ gives: $\mathcal{L}\{y\} = \frac{1}{(s-2)(s^2-1)} + \frac{1}{s^2-1}$.

   (c) *Partial Fractions*: $\frac{1}{(s-2)(s^2-1)} = \frac{1}{(s-2)(s-1)(s+1)} = \frac{A}{s-2} + \frac{B}{s-1} + \frac{C}{s+1}$ and you find $A = \frac{1}{3}$, $B = -\frac{1}{2}$, $C = \frac{1}{6}$.
   And $\frac{1}{s^2-1} = \frac{1}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1}$ and you find $A = \frac{1}{2}$ and $B = -\frac{1}{2}$.

   (d) *Inverse Laplace transform*: The solution is: $y(t) = \mathcal{L}^{-1}\{\frac{1/3}{s-2} - \frac{1/2}{s-1} + \frac{1/6}{s+1} + \frac{1/2}{s-1} - \frac{1/2}{s+1}\} = 2e^{2t} + \frac{1}{6}e^{-t} - \frac{1}{2}e^{-t}$.
   Thus, $y(t) = \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t}$.

4. Solve $y'' + y = u_5(t)$ with $y(0) = 0$ and $y'(0) = 3$, using the Laplace transform.

   (a) *Laplace Transform*: $\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{u_5(t)\}$.

   (b) *Use Rules and Solve*: $s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \mathcal{L}\{y\} = \frac{e^{-5s}}{s}$, which becomes: $(s^2 + 1)\mathcal{L}\{y\} - 3 = \frac{e^{-5s}}{s}$. Solving for $\mathcal{L}\{y\}$ gives: $\mathcal{L}\{y\} = \frac{e^{-5s}}{s(s^2+1)} + \frac{3}{s^2+1}$.

   (c) *Partial Fractions*: $\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$ and you find $A = 1$, $B = -1$, and $C = 0$.

   (d) *Inverse Laplace transform*: The solution is: $y(t) = \mathcal{L}^{-1}\{e^{-5s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + 3\frac{1}{s^2+1}\}$, which is the same as $u_5(t)\mathcal{L}^{-1}\{\frac{1}{s} - \frac{s}{s^2+1}\}(t-5) + 3\mathcal{L}^{-1}\{\frac{1}{s^2+1}\}(t)$, and we get $y(t) = u_5(t)(1 - \cos(t-5)) + 3\sin(t)$.