Skills Review: Complex Numbers

The following three pages give a quick introduction to complex numbers. The first page introduces basic arithmetic, the second page introduces Euler's formula, and the third page gives a graphical interpretation of complex numbers.

Introduction:
We define i to be a symbol that satisfies $i^2 = -1$. In other words, we think of i as a solution to $x^2 = -1$. The symbol i is called the **imaginary unit**.
Terminology:

- A **complex number** is any number that is written in the form $a + bi$ where a and b are real numbers.
- If $z = a + bi$ is a complex number, we say $\text{Re}(z) = a$ is the **real part** of the complex number and we say $\text{Im}(z) = b$ is the **imaginary part** of the complex number.

Basic Arithmetic:

1. We define all the same arithmetic properties. In other words, do arithmetic like you have always done. Just always replace i^2 by -1.
2. Here are several examples:
   - Adding Example: $(2 - 4i) + (10 + 7i) = 12 + 3i$.
   - Subtracting Example: $(-1 + 3i) - (4 - 5i) = -5 + 8i$.
   - Multiplying Example: $(3 + 2i)(5 - i) = 15 + 10i - 2i - 2i^2 = 17 + 8i$.
   - Powers Example: $i^3 = i^2 i = -i$, $i^4 = i^2 i^2 = (-1)(-1) = 1$, ...
3. Dividing: In order to divide you need to use the concept of the **conjugate**.
   The conjugate of $a + bi$ is $a - bi$.
   If you multiply a complex number by its conjugate you get $(a+bi)(a-bi) = a^2 - abi + abi - b^2 i^2 = a^2 + b^2$. When you divide by a complex number you should multiply the top and bottom by the conjugate of the denominator.
   Example: Simplify $\frac{4+i}{2-3i}$
   Multiplying top and bottom by $2 + 3i$ gives $\frac{(4+i)(2+3i)}{4+9} = \frac{8 + 2i + 12i - 3}{13} = \frac{5}{13} + \frac{14}{13}i$.
4. Other Powers: $e^{a+bi} = e^a e^{bi}$, $2^{a+bi} = 2^a 2^{bi} = 2^a e^{b\ln(2)i}$. For what do to with e^{bi} see the next page.

Solving Polynomial Equations (For your own interest):
Every solution to a polynomial equation is a real number or a complex number. (This is a part of what is called the fundamental theorem of algebra).
For example, $x^3 + 4x = 0$ has 3 solutions. Solving gives $x(x^2 + 2) = 0$, so $x = 0$ or $x = -2i$ or $x = 2i$.
Another example, if you ask Mathematica to solve $x^6 - 3x^2 + x = -10$, you get the six complex solutions:
$x \approx -1.26 - 0.53i$, $x \approx -1.26 + 0.53i$, $x \approx -0.03 - 1.62i$,
$x \approx -0.03 + 1.62i$, $x \approx 1.29 - 0.61i$, $x \approx 1.29 + 0.61i$.
(Notice the 6th power and the 6 solutions, that is not a coincidence, it is another part of the fundamental theorem of algebra).
As you see, complex numbers play a fundamental role in studying solutions to equations in algebra.

Euler's Formula

Euler's formula defines $e^{bi} = \cos(b) + i\sin(b)$.

For example:

- $e^{\frac{\pi}{6}i} = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}i.$
- $e^{5-\frac{\pi}{2}i} = e^5 e^{-\frac{\pi}{2}i} = e^5\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) = e^5(0-i) = -e^5 i$
- $e^{1+\frac{\pi}{4}i} = e e^{\frac{\pi}{4}i} = e\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = e\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \frac{\sqrt{2}e}{2} + \frac{\sqrt{2}e}{2}i$
- $e^{\pi i} = \cos(\pi) + i\sin(\pi) = -1$

This definition may seem odd at first, but, after you study Taylor series (in Math 126), you see that these do indeed give the same function. For those of you that have seen Taylor series, here is the Taylor series derivation of Euler's formula.

1. The Taylor series for e^z based at 0 is $e^z = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}z^n = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots$.
2. The Taylor series for $\sin z$ based at 0 is $\sin(z) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}z^{2n+1} = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots$.
3. The Taylor series for $\cos z$ based at 0 is $\cos(z) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}z^{2n} = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \cdots$.
4. Recognize that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$, $i^7 = -i$, $i^8 = 1$, ...
5. Now consider e^{bi}:

$$\begin{aligned}
e^{bi} &= 1 + bi + \frac{1}{2!}b^2i^2 + \frac{1}{3!}b^3i^3 + \frac{1}{4!}b^4i^4 + \frac{1}{5!}b^5i^5 + \frac{1}{6!}b^6i^6 + \cdots \\
&= 1 + bi - \frac{1}{2!}b^2 - \frac{1}{3!}b^3i + \frac{1}{4!}b^4 + \frac{1}{5!}b^5i - \frac{1}{6!}b^6 + \cdots \\
&= (1 - \frac{1}{2!}b^2 + \frac{1}{4!}b^4 - \cdots) + i(b - \frac{1}{3!}b^3 + \frac{1}{5!}b^5 + \cdots) \\
&= \cos(b) + i\sin(b)
\end{aligned}$$

Geometric Interpretations of Complex Numbers

Complex numbers $a + bi$ are often plotted on the xy-plane, where we take $x = a$ and $y = b$. When we plot complex numbers in this way, we say the xy-plane is the **complex plane**. We say the x-axis is the **real axis** and the y-axis is the **complex axis**.

1. Polar Coordinates: Consider a point (x, y) in the plane. Draw a line segment from the origin to the point (x, y). Label the length of this line segment r. Label the angle the line makes with the positive x-axis with the symbol θ. Using basic facts from trigonometry you get

$$x = r\cos(\theta)\ ,\ y = r\sin(\theta)\ ,\ x^2 + y^2 = r^2\ ,\ \tan(\theta) = \frac{y}{x}$$

   When we think of points in the plane in terms of r and θ, we say we are using polar coordinates.

2. Now, assume $a + bi$ is a complex number and write $a = r\cos(\theta)$ and $b = r\sin(\theta)$. Then we have

$$a + bi = r\cos(\theta) + ir\sin(\theta) = r(\cos(\theta) + i\sin(\theta)) = re^{\theta i}$$

3. Multiplying a complex number by $e^{\theta i}$ gives a new complex number that has been rotated counterclockwise by the angle θ.
   Here are several examples:

   - Consider the point $(x, y) = (0, 5)$ written as the complex number $z = 0 + 5i$.
     Multiplying by $e^{\frac{\pi}{2}i} = \cos(\pi/2) + i\sin(\pi/2) = i$ leads to counterclockwise rotation by 90 degree. Here is the multiplication $(0 + 5i)i = -5 + 0i$ which gives the new point $(-5, 0)$.
   - Consider the point $(x, y) = (2, 1)$ written as the complex number $z = 2 + i$.
     Multiplying by $e^{\frac{\pi}{i}} = \cos(\pi) + i\sin(\pi) = -1$ leads to a counterclockwise rotation by 180 degrees. Here is the multiplication $(2+i)(-1) = -2 - i$ which gives the new point $(-2, -1)$.
   - Consider the point $(x, y) = (-3, 4)$ written as the complex number $z = -3 + 4i$.
     Multiplying by $e^{\frac{\pi}{4}i} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ leads to a counterclockwise rotation by 45 degrees.
     Here is the multiplication $(-3+4i)\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = (\frac{-3\sqrt{2}}{2} - \frac{4\sqrt{2}}{2}) + (\frac{-3\sqrt{2}}{2} + \frac{4\sqrt{2}}{2})i = \frac{-7\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ which gives the new point $\left(\frac{-7\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.