Name and section: ______________________________

Solution: KEY!! Do not pass out!

Directions:

- You have 50 minutes to complete this exam.
- You are allowed a non-graphing calculator and a sheet of notes.
- Show all of your work, and put a box around your $\boxed{\text{final answer}}$.
- If you need more room, use the backs of the pages, and clearly indicate that you have done so.
- If you have any questions, raise your hand.

Question	Points	Score
1	6	
2	8	
3	10	
4	10	
5	8	
Total:	42	

1. (6 points) A spring-mass system has a spring constant of $k = 4$ N/m. A mass of 6 kg is attached to the spring. Let $\gamma = 5$ be the damping constant of the system.

   (a) What is the natural frequency of the system?

   Solution: $\sqrt{k/m} = \sqrt{4/6}$

   (b) Is the system over-damped, critically-damped, or neither? If neither, find the quasi-frequency of the system.

   Solution: $\gamma = 5 < 2\sqrt{4 \cdot 6} = 2\sqrt{km}$, so the answer is *neither*. Recall that in this case, the quasifrequency is μ where $r = \lambda + \mu i$. Notice that

   $$6u'' + 5u' + 4u' = 0.$$

   We solve the characteristic equation to see that

   $$\mu = \sqrt{4 \cdot 6 \cdot 4 - 25}/2 = \sqrt{71}/2.$$

   (c) Suppose we apply an external force $F(t) = 4\cos(\omega t)$ N. What is the resonant frequency (ω_{max}) of this forced system?

   Solution:

   $$\omega_{max} = \sqrt{\frac{k}{m} - \frac{\gamma^2}{2m^2}} = \sqrt{\frac{4}{6} - \frac{5^2}{2 \cdot 6^2}} \text{ or}$$
   $$\omega_0\sqrt{1 - \frac{\gamma^2}{2mk}} = \sqrt{\frac{4}{6}}\sqrt{1 - \frac{5^2}{2 \cdot 4 \cdot 6}}$$

   (d)

2. (8 points) A 64 lb object stretches a spring 64/9 feet. There is a damper with damping constant $\gamma = 8$ lb s/ft. The object is pulled down 1 foot and released. Use $g = 32$ ft/s as your gravitational constant.

   (a) Write down a differential equation for the motion of the spring. Include initial values.

   Solution: Find $m = 64/32 = 2, \gamma = 8$, and $k = 64/(64/9) = 9$. The differential equation is

   $$2u'' + 8u' + 9u = 0; \quad u(0) = 1, u'(0) = 0.$$

   (b) Use the equation you wrote above to find an equation for the motion of the spring.

   Solution: The roots of the characteristic equation are $-2 \pm i\sqrt{2}/2$. The general solution is

   $$e^{-2t}[c_1 \cos(\sqrt{2}t/2) + c_2 \sin(\sqrt{2}t/2)].$$

   Using the initial condition $u(0) = 1$, we get that $c_1 = 1$. Then using the condition $u'(0) = 0$, we find that $c_2 = 2\sqrt{2}$. So our final solution is

   $$u = e^{-2t}[\cos(\sqrt{2}t/2) + (2\sqrt{2})\sin(\sqrt{2}t/2)].$$

3. (10 points) (a) Give the general solution to the following differential equation.

$$y'' - 4y' - 5y = e^{-t},$$

Solution:
1) Solve the corresponding homogeneous equation $y'' + 4y' + -5y = 0$.
The characteristic equation is $r^2+4r-5=0$. Factoring gives $(r-5)(r+1)=0$, so $r=5$ or -1.
Thus $y_c = c_1e^{-t} + c_2e^{5t}$.
2) Find a particular solution. We could guess $Y = Ae^{-t}$, but this is a solution to the corresponding homogeneous equation, so we know it won't work. Instead try

$$Y = Ate^{-t}.$$

Then $Y' = Ae^{-t}(1-t)$ and $Y'' = Ae^{-t}(t-2)$.
Plug in to get $Ae^{-t}(t-2) + 4Ae^{-t}(1-t) - 5Ae^{-t} = e^{-t}$.
Simplify to $A(-6) = 1$, so $A = -\frac{1}{6}$, $Y = -\frac{1}{6}te^{-t}$.
3) Add them together to get $y = c_1e^{-t} + c_2e^{t} - \frac{1}{6}te^{-t}$.

(b) What family would you guess to find a particular solution (do not solve).

$$9y'' - 3y' - 7y = e^{2t}\cos(3t),$$

Solution:

$$Y = Ae^{2t}\cos(3t) + Be^{2t}\sin(3t)$$

4. (10 points) Consider the differential equation $t^2y'' + 3ty' - 8y = 0,\ t > 0$

(a) Verify that $y_1 = t^2$ is a solution.

Solution: First find $y_1' = 2t$ and $y'' = 2$. Now plug in to the left side and simplify:
$t^2(2) + 3t(2t) - 8(t^2) = 2t^2 + 6t^2 - 8t^2 = 0.$
Thus, y_1 is a particular solution.

(b) Find the general solution for y.

Solution: Use reduction of order by assuming $y = y_1 v$ for some function $v(t)$. Plugging in to the original equation and simplifying gives

$$t^2v'' + (2 \cdot 2t + \frac{3}{t}t^2)v' = 0$$

$$v'' + 7t^{-1}v' = 0$$

This is a second order differential equation in v'. Solving, we have

$$\ln(|v'|) = -7\ln(|t|) + c$$

$$v' = c_1t^{-7}$$

$$v = c_1t^{-6} + c_2$$

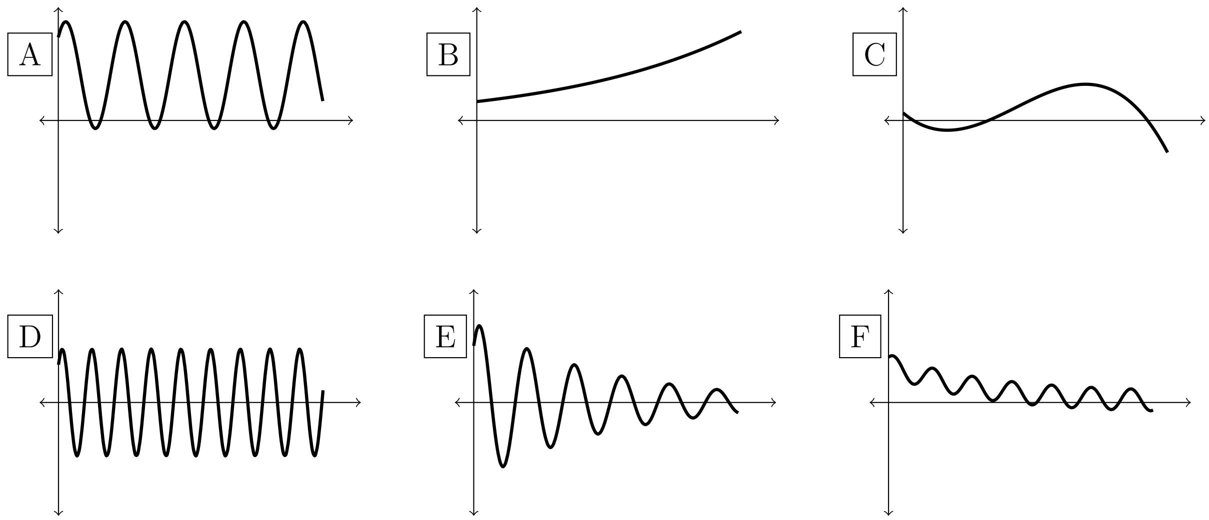
$$y = y_1v = c_1t^{-4} + c_2t^2$$

(c) Use a Wronskian to verify that the the solution you got is the general solution. (If you were unable to find a general solution, then make one up.)

Solution: Let $y_2 = t^{-4}$

$$W = y_1'y_2 - y_1y_2' = 2t \cdot t^{-4} + t^2 \cdot 4t^{-5} = 6t^{-3} \neq 0 \text{ when } t \neq 0.$$

5. (8 points) (a) Below are four graphs of functions $y(t)$. The t and y axes are shown.



For each equation below, figure out what its solutions would look like. If one of the graphs above could be a solution, write its label in the blank space. If none of the graphs could possibly be a solution, write “NONE”. Some graphs might not be used, and some might be repeated. No work necessary.

$y'' + 4y = 0$	D	$y'' + 4y' + 3y = 0$	NONE
$y'' - 4y' + 2y = 0$	B	$y'' + 2y' + 2y = 0$	E

Give an example of each of the following. No justification necessary.

(b) A second order nonlinear differential equation.

Solution: There are many examples of answers. Here is one: $y''y' = 1$.

(c) A second order differential equation for a function $u(t)$ such that $\lim_{t\to\infty} u(t) = 0$.

Solution: $u'' + u' + u = 0$