Math 300 Assignment 5

PROBLEMS: 4.45, 4.47, 5.6, 5.8, 5.30, 5.50, 6.8, 6.11, 6.17, 6.22, 6.24

DR. LOVELESS PROBLEMS

PROBLEM I: Using results from class give a short justification that $\mathbb{R} - \mathbb{Q}$ (the set of irrational numbers) is uncountable? Thus, in the sense of cardinality, the irrational numbers form a larger order of infinity than the rational numbers.

PROBLEM II: The two sums $A = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$ and $B = \sum_{k=0}^{n} \binom{n}{k} w^k (1-w)^{n-k}$ each give a fixed constant for all choices of $n \geq 1$ and w. Give the two constant values A and B.

PROBLEM III: Let $2 \leq k \leq n$. Prove that $\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}$.

PROBLEM IV: Let $n \geq 1$. The first part below is for practice with factorials, see if you can manipulate the factorials, in two different ways, in the definition of $2n$ choose n to get the two expressions. The second part follows directly from part (a), try to explain how. (The second part will reappear in a problem on the last homework, so make a mental note of this fact).

1. Explain why the three expressions below are equal:
$$\binom{2n}{n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{1 \cdot 2 \cdot 3 \cdots n} 2^n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} 2^{2n}.$$
2. Prove that $2^n \leq \binom{2n}{n} < 2^{2n}$.

The problems above are DUE FRIDAY, FEBRUARY 18th at lecture or during office hours.

HOMEWORK NOTES/HINTS

- PROBLEM 4.45: Give me a structure proof, but it is okay if your proof is a little informal (meaning you can use phrases like “all elements are hit” or “some elements are missed”). Try the contrapositive or contradiction for each direction. For one direction, you may want to use the so-called Pigeonhole Principle that states “If m objects are being placed into n classes and $m > n$, then some class must contain two or more objects.” That is, a function between finite sets A and B with $|A| > |B|$ must hit some element of B at least 2 times (by the Pigeonhole Principle). To show that the equivalence fails for infinite sets, give me an example where the theorem is not true for an infinite set.
- PROBLEM 4.47: Give formal proofs, that is give bijections f : even natural numbers $\to \mathbb{N}$ and g : odd natural numbers $\to \mathbb{N}$.
- PROBLEM 5.6: No proof required. Count the bijections from $\{1, 2\}$ to $\{1, 2\}$, then count the bijections from $\{1, 2, 3\}$ to $\{1, 2, 3\}$, etc. When you are confident you know the formula for the number of bijections from $\{1, 2, ..., n\}$ to $\{1, 2, ..., n\}$ then write it down.

- PROBLEM 6.8: I want to see you do the steps of the Euclidean algorithm (we won't get to the "integer" combination concepts until after the homework is due, so ignore that part)
- PROBLEMS 6.17: use the function $D(a, b)$ discussed in class.
- PROBLEM 6.24: Use induction on n. The phrase "3 divides $4^n - 1$ for every positive integer n" is defined by "for every positive integer n, there is an integer m such that $4^n - 1 = 3m$". Similarly, the phrase "6 divides $n^3 + 5n$ for every positive integer n" is defined as "for every positive integer n, there is an integer m such that $n^3 + 5n = 6m$".
- **REMINDERS**: Proofread your work to make sure it is organized and don't forget to answer all parts of each question. Now that we are several weeks into the quarter, I expect the proofs to be nicely written in paragraph form using complete sentences.
- If you finish the homework early or if you are looking for some extra practice try the following problems:

  CHALLENGE PROBLEMS: 4.48(especially challenging), 5.57, 12.22

  These are not due, but I will award at most 1 point of extra credit per challenge problem correctly completed.