Math 300 Final Exam Review Checklist

The final exam in Math 300 is comprehensive.

1. **Structure of Proofs**: This course is about mathematical proofs. Thus, most importantly you need to know all of the following techniques well:
   (a) Direct Proof (in particular, proofs from definitions and proof by looking at cases)
   (b) Proof by Contrapositive
   (c) Proof by Contradiction
   (d) Proof by Induction

2. **Fundamental Proof Tools**: In order to be able to do proofs you need to have a solid understanding of statements, negations and definitions. In particular you must know:
   (a) Negations of statements (including the negation of an implication, the negation of existential and universal quantifiers, and de'Morgan's laws).
   (b) Definitions of set union, intersection, difference, and complement (and how to use them in a subset proof).
   (c) Facts about inequalities (AGM, Triangle inequality, basic axioms)

3. **Specific Topics**:
   (a) Chapter 4: Know the definitions involving functions which include bounded, composition, one-to-one (injection), onto (surjection), bijection, and monotone (and the many variants of monotonicity). For all of these, understand the importance of the order and presentation of your proof.

   Also, understand the definitions of finite, countable, or uncountable sets and be able to classify the standard sets (reals, rationals, integers, natural numbers).

   (b) Chapter 5: Know the binomial theorem and Pascal's formula and be able to recognize when they can and should be used.
   (c) Chapter 6: Know the definition of $a \mid b$ and $\gcd(a, b)$ and how to use these definitions in proofs. There were a lot of different things we proved about the $\gcd(a, b)$ which included
   (1) simplification in divisibility problems (see 'Important Result 2' in the Ch. 6 Review) and
   (2) when the equation $ax + by = c$ has integer solutions.
   These are important facts to know in order to be able to give proofs about divisibility.

   Understand what it means for a number to be a prime and know what the fundamental theorem of arithmetic is saying.

   Know the division algorithm and the Euclidean algorithm and how they are used.

   (d) Chapter 7: Know the definition of $a \equiv b \pmod{n}$ and all the facts that we know about simplifying congruences and using them in proofs. In particular, you should know how to translate a divisibility problem into a congruence problem and you should understand how every integer fits into one of n congruence classes modulo n.

   You should also know the properties of an equivalence relation.

Some Closing Comments

Mathematics is a long journey that requires patience

To emphasize this fact, let me a share quote by a well-known mathematician, Andrew Wiles, that discusses how mathematics is done. A famous conjecture (Fermat's Theorem) states: If an integer n is greater than 2, then the diophantine equation $x^n + y^n = z^n$ has no solutions in non-zero integers x, y, and z. For example, $x^2 + y^2 = z^2$ is possible with integers (namely $3^2 + 4^2 = 5^2$). However, it is impossible to find non-zero integers that work for any integer exponent, n, greater than 2. This was conjectured by Fermat in 1637 and stood as one of the greatest unsolved problems in mathematics until 1993 when it was finally solved by Andrew Wiles (his final revised and reviewed proof was given in 1995). When discussing the process he went through to arrive at his proof he gave the following quote which gives a nice insight into how mathematics is really done (I think you can relate to this quote on a smaller time scale):

> *Perhaps I can best describe my experience of doing mathematics in terms of a journey through a dark unexplored mansion. You enter the first room of the mansion and it's completely dark. You stumble around bumping into the furniture, but gradually you learn where each piece of furniture is. Finally, after six months or so, you find the light switch, you turn it on, and suddenly it's all illuminated. You can see exactly where you were. Then you move into the next room and spend another six months in the dark. So each of these breakthroughs, while sometimes they're momentary, sometimes over a period of a day or two, they are the culmination of-and couldn't exist without-the many months of stumbling around in the dark that proceed them.* - Andrew Wiles

About Mathematics

I hope this quarter that you have gotten a glimpse of what mathematics is all about. The goal in mathematics is to discover new theorems and prove why they always work. Much care, determination, time and precision goes into forming a proof. But once a correct proof is given for a theorem, that theorem remains true forever. In this way mathematics is different than any other subject. It requires patience, creativity and an eye for detail.

The proof of a mathematical theorem can change the world. If you think I am overstating, you don't have to look any further than the electronic device in your pocket, the building you are in, the internet you log onto, or the bridges you use. Each of these required centuries of mathematical reasoning and proofs before they could be created, built, optimized and analyzed. What makes mathematics seductive is that significant theorems and proofs can be discovered by pure thought and pencil and paper.

For me, and many mathematicians, the joy of mathematics is not the applications (although they are a nice byproduct). The joy of mathematics is in the proofs themselves. Finding striking conclusions through basic reason is challenging and rewarding in itself. Putting it more simply, there is beauty in figuring things out. I hope you have gotten a taste of the beauty of mathematics this quarter. And if your mathematical journey continues, I promise it gets easier and significantly more interesting.

Further Study

For some fun and challenging proof problems that you can now understand check out the following books:

- <u>Proofs from THE BOOK</u>, by Martin Aigner and Gnter M. Ziegler.
- <u>Mathematical Gems I</u>, by Ross Honsberger.
- <u>Mathematical Gems II</u>, by Ross Honsberger.

For more about number theory try the books:

- Number Theory, by George Andrews (I original taught myself number theory from this book and greatly enjoyed it).
- A Friendly Introduction to Number Theory, by Joseph Silverman (A newer book with lots of tables and explanation).
- Cryptanalysis of Number Theoretic Ciphers, by Samuel Wagstaff (Contains an introduction to number theory, but also information about how it is used in cryptography)

For specific courses relating to concepts from this quarter try (note: some of these courses aren't offered every year, so you may have to pick and choose based on availability):

- To learn more about number theory, consider taking the course Math 301: Elementary to Number Theory and Math 414/415: Number Theory. There is also a lot of number theory in the course Math 411/412: Modern Algebra for Teachers and the courses 402/403/404: Abstract Algebra.
- To learn more about functions, real numbers, and the foundations of calculus take Math 327/328: Real Analysis or the year long series Math 424/425/426: Fundamental Concepts of Analysis (These are fundamental course that all mathematicians need in their background). And going further you should also take Math 427/428: Complex Analysis (which explores how things change when complex numbers are included).
- To learn more about the axioms of algebra and to further discuss additive and multiplicative inverses in an abstract setting take Math 402/403/404: Abstract Algebra (This is also a fundamental course that all mathematicians need in their background).
- To learn about combinatorics take Math 461/461: Combinatorial Theory or the applied course Math 381: Discrete Mathematical Modeling - this is a project based course where you would use discrete math (combinatorics, number theory, graph theory) to solve a real world problem.
- Some of the things we have mention (the binomial theorem, some number theory and certainly proof methods) will be helpful if you take probability/statistic courses including Math 390, 394, 395, 396.
- And, in general, proofs will be a major component of any upper level course even those classes that are more of an applied math nature including Math 407/408/409: Optimization, Math 435/436: Dynamical Systems (advanced differential equations), and Math 464/465/466: Numerical Analysis (approximation methods).
- If you are interested in generalizing concepts of real analysis (spaces, surfaces, higher dimensions) consider the courses Math 441: Topology and Math 442: Differential Geometry.
- There are many other topics of theoretic interest and applied interest and they all are interesting in their own way. Talk to classmates, or me, to find out what courses will be most interesting to you.

Other programs in the math department

- I strongly encourage you to get involved with the Putnam Exam. It is a very difficult exam that is given in December each year of which the problems are somewhat similar to challenge problems from our course. The exam consists of 2 parts, 5 questions each. The median exam score is ZERO. So even if you don't get any right, you still are doing as well as half the students. If you don't do well it doesn't hurt you at all. But if you get a high score, then graduate schools will try to recruit you to their programs. So it's a great opportunity for you. For more information, visit: www.math.washington.edu/ dumitriu/putpage.html (A couple professors in our department run a seminar to prepare students for the exam, the link above will explain this and it provides links to more information).

- I also strongly encourage you to get involved in the Mathematical Contest in Modeling which is a done in the spring of every year. You can read former contest entries at www.math.washington.edu/ morrow/mcm/mcm.html (For more information, visit the math student services office in Padelford C-36)

- Each spring there is a "Math Day" where high school students from around the state come to participate in math events and competitions. The department is always looking for volunteers to help out. It's a minimal commitment and it's fun.

- If you have any interest in undergraduate research, please visit the Math Student Services office in Padelford C-36 and speak with any of the secretaries. They can try to get you in an undergraduate research program and they can let you know about other opporunities.