Math 300 Chapter 5 Review

PERMUTATIONS

$$\begin{aligned} n! &= n(n-1)\cdots 2\cdot 1 = \textstyle\prod_{i=1}^{n} i \\ &= \text{the number of ways to arrange (in some order) all the elements of } \{1,2,\ldots,n\} \text{ without repeats} \\ &= \text{the number of permutations of a set of size } n. \end{aligned}$$

Example: The number of ways to permute the set $\{1,2,3\}$ is $3! = 3\cdot 2\cdot 1 = 6$.
In particular, the 6 permutations are (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), and (3,2,1).

$$\begin{aligned} \textstyle\prod_{i=n-k+1}^{n} i &= n(n-1)\cdots(n-k+1) \\ &= \text{the number of ways to arrange (in some order) } k \text{ elements of } \{1,2,\ldots,n\} \text{ without repeats.} \end{aligned}$$

Example: The number of ways to arrange 2 elements from the set $\{1,2,3,4\}$ is $4\cdot 3 = 12$.
In particular, the 12 arrangements are (1,2), (2,1), (1,3), (3,1), (1,4), (4,1), (2,3), (3,2), (2,4), (4,2), (3,4), and (4,3).

COMBINATIONS

$$\begin{aligned} \binom{n}{k} &= \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!} \\ &= \text{the number of ways to select (in no particular order) } k \text{ elements of } \{1,2,\ldots,n\} \text{ without repeats.} \end{aligned}$$

Example: The number of ways to select 2 elements from the set $\{1,2,3,4\}$ is $\binom{4}{2} = 6$.
In particular, the 6 selections are {1,2}, {1,3}, {1,4}, {2,3}, {2,4}, {3,4}.
The last two examples illustrate the fact that:
(number of ways to permute each selection) $\times$ (number of k-selections) = (number of k-arrangements).
That is, $k!\binom{n}{k} = n(n-1)\cdots(n-k+1)$.

FACTS ABOUT $\binom{n}{k}$

1. *The Binomial Theorem*: For all $x, y \in \mathbb{R}$,

$$\begin{aligned} (x+y)^n &= \textstyle\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\ &= \binom{n}{0} y^n + \binom{n}{1} xy^{n-1} + \binom{n}{2} x^2 y^{n-2} + \cdots + \binom{n}{n-1} x^{n-1} y + \binom{n}{n} x^n. \end{aligned}$$

2. *Pascal's Formula*: For $1 \le k \le n$,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

3. *Symmetry*: For $0 \le k \le n$,

$$\binom{n}{k} = \binom{n}{n-k}.$$