Math 300 Chapter 4 Overview

Functions/Injections/Surjections/Bijections/Inverses/Composition

Let $f : A \to B$ and $g : B \to C$ be functions.

- **function**: For f to be a function, it must satisfy (i) *FOR ALL* $a \in A$ there is a rule $f(a)$ which gives an element in B, (ii) f must be well-defined.

- **well-defined**: For all $x_1, x_2 \in A$, $x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$.

- **injective (one-to-one)**: For all $x_1, x_2 \in A$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

- **surjective (onto)**: For all $b \in B$, there exists $a \in A$ such that $f(a) = b$.

- **bijection**: f is both injective and surjective.

- **inverse**: If f is a bijection, then the inverse function of f exists and we write $f^{-1}(b) = a$ to means the same as $b = f(a)$.

- **composition**: The function $h = g \circ f : A \to C$ is called the composition and is given by $h(x) = g(f(x))$ for all $x \in A$.

- To prove that f is an injection (one-to-one).

  1. Assume $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$.
  2. Use whatever is given to deduce that $x_1 = x_2$.

- To prove that f is a surjection (onto).

  1. Assume $b \in B$.
  2. Use whatever is given to find a solution that will work for $f(a) = b$ for some $a \in A$. Sometimes this simply involves observing a pattern for the input and using it to say what a needs to be. Other times, you will need to solve or use other functions. In any case, you are trying to show that b can always be obtained by plugging 'something' into f (you need to show what this 'something' should look like).

Notes:

- When giving examples involving a function f, make sure that you are giving a rule for ALL elements of A.

- It is very important that you pay attention to the domain and range. Changing the domain or range often changes the properties of the function f.

- The inverse function and the inverse notation only applies if f is a bijection. Otherwise the inverse function does not make sense.

- (Injection/Surjection and Compositions) The composition function $g \circ f$ does not require information about B. So we can't tell if f is onto (because we don't know if the elements of B were all hit or not) and we don't know if g is one-to-one (because we don't know where g is mapping from). Here is what we do know:

  1. f and g both one-to-one $\Rightarrow g \circ f$ is one-to-one.
  2. f and g both onto $\Rightarrow g \circ f$ is onto.
  3. f and g both bijections $\Rightarrow g \circ f$ is a bijection and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
  4. Problem 4.34 discusses what you can say in the converse direction.

Increasing/Decreasing/Monotone

Let $f : A \to B$ be a function with $A, B \subseteq \mathbb{R}$.

- **increasing**: For all $x_1, x_2 \in A$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

- **decreasing**: For all $x_1, x_2 \in A$, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

- **nondecreasing ('increasing or flat')**: For all $x_1, x_2 \in A$, $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$.

- **nonincreasing ('decreasing or flat')**: For all $x_1, x_2 \in A$, $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$.

- **monotone**: f is nondecreasing or nonincreasing over all A.

- **strictly monotone**: f is increasing or decreasing over all A.

Notes:

- Many functions do not satisfy any of these definitions!

- In order for any of these definitions to hold, the statement must be true '*FOR ALL* $x_1, x_2 \in A$ such that $x_1 < x_2$'. If it only works for some x_1 and x_2, then the function is not necessarily monotone.

- To prove any of these, you start by letting $x_1, x_2 \in A$ such that $x_1 < x_2$, then you try to deduce what happens to the inequality between $f(x_1)$ and $f(x_2)$.

- *Increasing and nondecreasing*, essentially are occurring when the inequality is always in the *same direction* as x_1 and x_2.

- *Decreasing and nonincreasing*, essentially are occurring when the inequality is always in the *opposite direction* as x_1 and x_2.

- (Injection/Surjection/Compositions and Monotone): Make sure you understand all the proofs and counterexamples that appeared in the problems in your homework. The homework shows you that you have be careful about what you assume and the problems show you the importance of understanding the definitions. However, we found that a few general results are true. Here is a summary of the true results:

    1. f strictly monotone $\Rightarrow f$ is one-to-one. (you can't go the other direction)
    2. f, g monotone $\Rightarrow g \circ f$ is monotone.
    3. Let f be a bijection. f is monotone if and only if f^{-1} is monotone.
    4. $f : \mathbb{R} \to \mathbb{R}$ onto $\Rightarrow f$ is unbounded.

Cardinality

- **Notation:** $[k] = \{1, 2, ..., k\}$, and $[0] = \emptyset$.

- **Finite Sets:** The set A is **finite with cardinality** $|A| = n$ if there exists a bijection $f : A \to [n]$ (or a bijection $g : [n] \to A$). If no such bijection exists (for any integer $n \geq 0$), then we say that A is **infinite**.

- **Countably Infinite Sets:** The set A is **countable** if there exists a bijection $f : A \to \mathbb{N}$ (or a bijection $g : \mathbb{N} \to A$).

- **Uncountably Infinite Sets:** If A is an infinite set and A is not countable, then we say A is uncountable.

- **Cardinality:** The sets A and B have the same cardinality if and only if there exists a bijection $f : A \to B$.

Let me say the last comment above again since it is the essence of cardinality!
We say A and B have the same cardinality (*i.e.* $|A| = |B|$) iff there exists a bijection $f : A \to B$.

Thus, to formally prove $|A| = |B|$, we must
(1) Give a specific function $f : A \to B$.
(2) Prove that the function we gave is a bijection (one-to-one and onto).

Notes:

- Let A and B be finite sets.

    1. If $f : A \to B$ is one-to-one, then $|A| \leq |B|$.
    2. If $f : A \to B$ is onto, then $|A| \geq |B|$.
    3. If $f : A \to B$ is a bijection, then $|A| = |B|$.

- We discussed the following results about countable sets:

    1. If A_1 and A_2 are countable, then $A_1 \cup A_2$ is countable.
    2. More generally, if $A_1, A_2, A_3, \ldots$ is a sequence of countable sets, then $A = A_1 \cup A_2 \cup \cdots$ is countable.
    3. If A is countable and $S \subseteq A$ is an infinite subset, then S is countable.

- The contrapositive of the results above can be used to make statements about uncountable sets. Here are the contrapositives:

    1. If $A_1 \cup A_2$ is uncountable, then A_1 or A_2 is uncountable.
    2. If $A = A_1 \cup A_2 \cup \cdots$ is uncountable, then at least one of A_1 or A_2 or A_3 or $\ldots$ is uncountable.
    3. If S is uncountable and $S \subseteq A$, then A is uncountable.

- We also proved the special cases below:

    1. $\mathbb{N}$, $\mathbb{N} \times \mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are all countable.
    2. $(0, 1)$ and $\mathbb{R}$ are both uncountable.