

Math 300 Definitions and Theorems from Chapter 1

Essential Definitions

1. “If P , then Q ” is the same as “ P implies Q ” which can be written as “ $P \Rightarrow Q$ ”.
2. A **counterexample** to $P \Rightarrow Q$ is an example where P is true and Q is false.
3. “ P if and only if Q ” means both “ $P \Rightarrow Q$ and $Q \Rightarrow P$ ” which can be written as “ $P \Leftrightarrow Q$ ” (This means that P and Q are logically equivalent and we can go back and forth between them).
4. $n \in \mathbb{Z}$ is even $\Leftrightarrow n = 2k$ for some $k \in \mathbb{Z}$.
5. $n \in \mathbb{Z}$ is odd $\Leftrightarrow n = 2k + 1$ for some $k \in \mathbb{Z}$.
6. $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$. (meaning either $x \in A$, $x \in B$, or both)
7. $x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$.
8. $x \in A - B \Leftrightarrow x \in A$ and $x \notin B$.
9. $x \in A^c \Leftrightarrow x \notin A$.
10. $A \subseteq B \Leftrightarrow (x \in A \text{ implies } x \in B)$.
11. $A \subset B \Leftrightarrow (x \in A \text{ implies } x \in B)$ and there is at least one element in B that is not in A (this is the definition of **proper subset**).
12. $f : A \rightarrow B$ is a **function** $\Leftrightarrow f$ assigns to each element $a \in A$ a single element $b = f(a)$ in B . A is called the **domain** and B is call the **target** (or **range**). It is important to note that a function f , by definition, must have a rule to map **each element** of A . For instance $f(x) = 1/x$ is not a function from $f : \mathbb{R} \rightarrow \mathbb{R}$ because it doesn't tell you where the number zero will be mapped (it is a function $f : \mathbb{R} - \{0\} \rightarrow \mathbb{R}$). It is okay if not everything in the target set B is mapped to (We will talk a lot about properties of functions in Chapter 4).
13. $y \in f(S) \Leftrightarrow$ there exists an $x \in S$ such that $y = f(x)$.
14. f is bounded \Leftrightarrow there exists an $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$.

Often Used Axioms and Theorems

1. (Triangle Inequality) $\forall x, y \in \mathbb{R}, |x + y| \leq |x| + |y|$.
2. (AGM Inequality) $\forall x, y \in \mathbb{R}$,
 - (a) $2xy \leq x^2 + y^2$,
 - (b) $xy \leq \left(\frac{x+y}{2}\right)^2$, and
 - (c) if $x, y \geq 0$, then $\sqrt{xy} \leq \frac{x+y}{2}$.
3. (Axioms)
 - If $x \leq y$ and $u \leq v$, then $x + u \leq y + v$. (*i.e.* we can add inequalities).
 - If $0 \leq x \leq y$ and $0 \leq u \leq v$, then $xu \leq yv$. (*i.e.* we can multiply inequalities provided all values are positive. Note: When we work with negative numbers, multiplication and inequalities we must be careful about when to switch the direction of the inequality.)
 - $|x|^2 = x^2$ for all $x \in \mathbb{R}$.
 - $x^2 \geq 0$ for all $x \in \mathbb{R}$.

Direct Proof ($P \Rightarrow Q$) At this point in the term our only logical method of proof is direct proof (start from the hypothesis and end with the conclusion). This is what your proofs should look like:

Theorem: P implies Q .

proof: Assume P is true.

⋮

(Here you write out the hypotheses in P and try to show, using logical deductions based on known facts, that Q is true)

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Thus, Q is true. ■

Special Proofs

1. **Subset Proofs** To prove $\text{BLAH} \subseteq \text{STUFF}$: You must start with an arbitrary element of BLAH and show, from the conditions on BLAH, that the object must also be an element of STUFF. In other words you will prove “ $x \in \text{BLAH}$ implies $x \in \text{STUFF}$ ”. Thus, the first line of your proof will be “Assume $x \in \text{BLAH}$.” and the last line of your proof will be “Thus, $x \in \text{STUFF}$.”
2. **Set Equality Proofs** To prove two sets are equal ($A = B$). You must show two things, $A \subseteq B$ and $B \subseteq A$ (for each of these separately you will use the subset proof method given above).