Real Number Axioms and Elementary Consequences

As much as possible, in mathematics we base each field of study on axioms. Axioms are rules that give the fundamental properties and relationships between objects in our study. We do not prove axioms! We take them as mathematical facts and we deduce theorems from them. As long as the axioms don't contradict one another, we can make up any axioms we want to give structure to what we are studying. How and why we choose certain axioms is a more philosophical question than a mathematical one. For the real number system, the axioms are chosen so that the standard properties of numbers (that you are familiar with from grade school) hold.

Field Axioms

A *field* is a set, S, with operations '+' and '$\cdot$' such that

A0: $x + y \in S$	M0: $x \cdot y \in S$	Closure
A1: $(x + y) + z = x + (y + z)$	M1: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$	Associativity
A2: $x + y = y + x$	M2: $x \cdot y = y \cdot x$	Commutativity
A3: $x + 0 = x$	M3: $x \cdot 1 = x$	Identity
A4: Given $x \in S$, there is $(-x) \in S$ such that $x + (-x) = 0$	M4: For $x \neq 0$, there is $1/x \in S$ such that $x \cdot (1/x) = 1$	Inverse
DL: $x \cdot (y + z) = x \cdot y + x \cdot z$		Distribution

Notes on the Field Axioms:

1. These properties are satisfied by $\mathbb{Q}$ and $\mathbb{R}$ (and $\mathbb{Z}_p$ which we will study at the end of the term).
2. By closure, you can add or multiply any two elements of S as you wish and get another element of S.
3. An 'operation' must always output the same element when given the same inputs (this is called being 'well-defined'). That is,

$$\text{if } x = y \text{ and } z \text{ is any element in } S\text{, then } x + z = y + z \text{ and } x \cdot z = y \cdot z.$$

Order Axioms

A field, F, is *ordered* if it contains a positive set P such that: P1: $x, y \in P$ implies $x + y \in P$, P2: $x, y \in P$ implies $x \cdot y \in P$, and P3: $x \in F$ implies $x = 0$, $x \in P$, or $-x \in P$.

Notes on Order Axioms:

1. These properties are satisfied by $\mathbb{Q}$ and $\mathbb{R}$. Basically, this says that all numbers are positive, negative, or zero and that the sum or product of positive numbers is positive.
2. We define $x < y$ to mean that $y - x$ is positive (*i.e.* $y - x \in P$). And $x \leq y$ means $y - x \in P$ or $y - x = 0$.

Completeness Axiom

An ordered field F is *complete* if every nonempty subset of F that has an upper bound in F has a least upper bound in F.

The completeness axiom is satisfied by $\mathbb{R}$ and not by $\mathbb{Q}$. It basically says that there are 'no gaps' in the set. The set $\mathbb{R}$ can be classified as a complete ordered field.

What this means for you when doing proofs

In this class you may use any standard arithmetic manipulations from grade school without directly referencing the axiom they come from (if you want to emphasize that you are using a basic fact, then you can just say 'using algebra', 'by axioms', or just explain the arithmetic you are doing). The short list below includes the main properties that you will be using.

1. You can add, subtract, and multiply both sides of an equation by any number. (by A4 and definitions of the operations)
2. You can divide both sides of an equation by x only if $x \neq 0$. (by M4)
3. You can add or subtract both sides of an inequality by any number. (consequence of order axioms)
4. If you multiply, or divide (if not zero), both sides of an inequality by x, then
   (a) if $x \geq 0$, then the inequality remains in the same direction.
   (b) if $x < 0$, then the inequality switches direction.
5. You can add inequalities that are in the same direction. (See F3)
6. You can multiply inequalities only if they point in the same direction and all numbers are positive. (See F4)

These facts along with several others (including the basic inequality rules) can be found on pages 16 and 17 of your text. **Throughout the quarter you may assume all of these axioms and propositions without directly referencing them. However, all other facts in your proofs must be justified by either referencing a previous theorem or logical reasoning.**