Binomial Coefficients and Permutations Mini-lecture

The following pages discuss a few special integer counting functions. You may have seen some of these before in a basic probability class or elsewhere, but perhaps you haven't used them in full generality. These functions are incredibly useful and they are interesting to study in their own right. We will use each of these functions from time to time throughout the quarter.

Let's start with some definitions:

1. $n! = \prod_{i=1}^{n} i = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$
   = 'number of ways to arrange (permute) the entire set $[n] = \{1, 2, \ldots, n\}$ in some order'
   = 'number of ways to define a bijective function f from $[n]$ to $[n]$'
   = 'n factorial'

   By convention, we also define $0! = 1$.

   Examples:

   - $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$.
   - $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$
   - Note: $\frac{6!}{4!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{6 \cdot 5}{1} = 30$, so fractions involving factorials often have a lot of cancellation. For example, $\frac{(n+1)!}{n!} = \frac{(n+1) \cdot n \cdot (n-1) \cdots 2 \cdot 1}{n \cdot (n-1) \cdots 2 \cdot 1} = \frac{n+1}{1} = n+1$. This is useful to remember when working with factorials.

2. $P_{n,k} = n(n-1)(n-2) \cdots (n-k+1)$
   = 'number of ways to arrange (permute) k elements from the set $[n] = \{1, 2, \ldots, n\}$.'
   = 'number of ways to define an injective function from $[k]$ to $[n]$.' Examples:

   - $P_{6,3} = 6 \cdot 5 \cdot (6-3+1) = 6 \cdot 5 \cdot 4 = 120$.
   - $P_{6,4} = 6 \cdot 5 \cdot 4 \cdot (6-4+1) = 6 \cdot 5 \cdot 4 \cdot 3 = 360$.
   - In other words, $P_{n,k}$ is just like the factorial except you stop after multiplying k numbers starting at n.

3. $C_{n,k} = \binom{n}{k}$
   = 'number of ways to choose k elements from the set $[n] = \{1, 2, \ldots, n\}$ (order doesn't matter).'
   = 'n choose k.'

   The formulas for this are somewhat less direct, they all use the two previous functions. To itemize the ways to *choose* k items sometimes takes a bit longer. Examples will be given below.

   By convention, we also define $\binom{0}{0} = 1$ and if $k > n$ or $k < 0$ we define $\binom{n}{k} = 0$.

On the next page, we explore connections between these functions to get a formula for $\binom{n}{k}$.

The Connection Between $P_{n,k}$ and $C_{n,k}$

1. Consider the set $[4] = \{1, 2, 3, 4\}$ and let $k = 2$.

Ways to **choose** $k = 2$ elements from [4]	Ways to **arrange** $k = 2$ elements from [4]
$\{1, 2\}$	(1,2), (2,1)
$\{1, 3\}$	(1,3), (3,1)
$\{1, 4\}$	(1,4), (4,1)
$\{2, 3\}$	(2,3), (3,2)
$\{2, 4\}$	(2,4), (4,2)
$\{3, 4\}$	(3,4), (4,3)

   (a) Thus, the number of ways to choose $k = 2$ from $\{1, 2, 3, 4\}$ is $C_{4,2} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 6$.

   (b) The number of ways to arrange $k = 2$ elements from $\{1, 2, 3, 4\}$ is 12 which matches up with the formula: $P_{4,2} = n(n-1)\dots(n-k+1) = 4 \cdot 3 = 12$ (because $n - k + 1 = 3$).

   (c) NOTE THAT $P_{4,2} = 2 \cdot C_{4,2}$.

2. Consider the set $[5] = \{1, 2, 3, 4, 5\}$ and let $k = 3$.

Ways to **choose** $k = 3$ elements from [5]	Ways to **arrange** $k = 3$ elements from [5]
$\{1, 2, 3\}$	(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)
$\{1, 2, 4\}$	(1,2,4), (1,4,2), (2,1,4), (2,4,1), (4,1,2), (4,2,1)
$\{1, 2, 5\}$	(1,2,5), (1,5,2), (2,1,5), (2,5,1), (5,1,2), (5,2,1)
$\{1, 3, 4\}$	(1,3,4), (1,4,3), (3,1,4), (3,4,1), (4,1,3), (4,3,1)
$\{1, 3, 5\}$	(1,3,5), (1,5,3), (3,1,5), (3,5,1), (5,1,3), (5,3,1)
$\{1, 4, 5\}$	(1,4,5), (1,5,4), (4,1,5), (4,5,1), (5,1,4), (5,4,1)
$\{2, 3, 4\}$	(2,3,4), (2,4,3), (3,2,4), (3,4,2), (4,2,3), (4,3,2)
$\{2, 3, 5\}$	(2,3,5), (2,5,3), (3,2,5), (3,5,2), (5,2,3), (5,3,2)
$\{2, 4, 5\}$	(2,4,5), (2,5,4), (4,2,5), (4,5,2), (5,2,4), (5,4,2)
$\{3, 4, 5\}$	(3,4,5), (3,5,4), (4,3,5), (4,5,3), (5,3,4), (5,4,3)

   (a) Thus, the number of was to choose $k = 3$ from $\{1, 2, 3, 4, 5\}$ is $C_{5,3} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} = 10$.

   (b) The number of ways to arrange $k = 3$ elements from $\{1, 2, 3, 4, 5\}$ is 60 which matches up with the formula: $P_{5,3} = n(n-1)\dots(n-k+1) = 5 \cdot 4 \cdot 3 = 60$ (because $n - k + 1 = 3$).

   (c) NOTE THAT $P_{6,3} = 6 \cdot C_{6,3}$ AND $6 = 3!$.

3. In summary, we see that $P_{4,2} = 4 \cdot 3 = 2! \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 2! C_{4,2}$ and $P_{5,3} = 5 \cdot 4 \cdot 3 = 3! \begin{pmatrix} 5 \\ 3 \end{pmatrix} = 3! C_{5,3}$. In general, $P_{n,k} = k! C_{n,k}$, thus

$$n(n-1)(n-2)\cdots(n-k+1) = k! \begin{pmatrix} n \\ k \end{pmatrix}.$$

   Therefore, when we solve for $\begin{pmatrix} n \\ k \end{pmatrix}$ we get

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \text{ which is often written as } \begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}.$$

Here are a few examples where we use the formula from the previous page:

- $\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot3\cdot2\cdot1} = \frac{5\cdot4}{2} = 5\cdot2 = 10$. Note that $\binom{5}{3} = \binom{5}{2}$, this is not a coincidence, $\binom{n}{k} = \binom{n}{n-k}$ is always true.

- $\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot4\cdot3\cdot2\cdot1} = \frac{7\cdot6\cdot5}{3\cdot2\cdot1} = 7\cdot5 = 35$

- Note: $\frac{6!}{4!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1} = \frac{6\cdot5}{1} = 30$, so fractions involving factorials often have a lot of cancellation. For example, $\frac{(n+1)!}{n!} = \frac{(n+1)\cdot n\cdot(n-1)\cdots2\cdot1}{n\cdot(n-1)\cdots2\cdot1} = \frac{n+1}{1} = n+1$. This is useful to remember when working with factorials.

Pascal's Identity

The numbers $\binom{n}{k}$ have some nice properties if you list them out. Most notably is the recurrence formula attributed to Pascal. Here the first several values of $\binom{n}{k}$ are given (n is the row and k is the column). See if you can spot a relationship between the rows and columns, the answer immediately follows the table so try to find the pattern before you read on.

		k						
		0	**1**	**2**	**3**	**4**	**5**	**6**
	0	1						
	1	1	1					
n	**2**	1	2	1				
	3	1	3	3	1			
	4	1	4	6	4	1		
	5	1	5	10	10	5	1	
	6	1	6	15	20	15	6	1

The relationship is that each entry is the sum of the two entries immediately above and immediately above and to the left. For example, $\binom{5}{3} = \binom{4}{3} + \binom{4}{2}$. This relationship is given in the following theorem.

Pascal's Identity: For all $n, k \in \mathbb{Z}$ such that $0 < k \leq n$,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

The Binomial Theorem

The main reason we are discussing these functions is because of their appearance in the binomial theorem. When we expand out a binomial to various powers a pattern appears. For example (I encourage you to check these expansions by doing the multiplication out the long way).

- $(x+y)^2 = x^2 + 2xy + y^2$.

- $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.

- $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.

There are familiar numbers. In fact the coefficient of $x^n y^{n-k}$ is the expansion of $(x+y)^n$ is always $\left(\begin{array}{c} n \\ k \end{array}\right)$, for this reason the numbers $\left(\begin{array}{c} n \\ k \end{array}\right)$ as often referred to as binomial coefficients. So we have the following theorem:

The Binomial Theorem: For all $n \in \mathbb{Z}$ and for all $x, y \in \mathbb{R}$,

$$(x+y)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) x^{n-k} y^k$$

$$= \left(\begin{array}{c} n \\ 0 \end{array}\right) x^n y^0 + \left(\begin{array}{c} n \\ 1 \end{array}\right) x^{n-1} y^1 + \left(\begin{array}{c} n \\ 2 \end{array}\right) x^{n-2} y^2 + \cdots + \left(\begin{array}{c} n \\ n-1 \end{array}\right) x^1 y^{n-1} + \left(\begin{array}{c} n \\ n \end{array}\right) x^0 y^n.$$

For example,

$$\begin{aligned} (x+y)^6 &= \textstyle\sum_{k=0}^{6} \left(\begin{array}{c} 6 \\ k \end{array}\right) x^{6-k} y^k \\ &= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6 \end{aligned}$$

Practice Problems

1. Prove that $\binom{n}{k} = \binom{n}{n-k}$. (Hint: The proof will be short, you can either use a combinatorial argument for why they would give the same number or use the known formula for binomial coefficients.)

2. Prove that if $\binom{n}{k}$ and $\binom{n}{k-1}$ are both even, then $\binom{n+1}{k}$ is even. (So the number 'below' two even numbers in Pascal's triangle is also even).

3. (a) Using the binomial theorem, tell me what needs to go in place of the question marks in

$$1 + \binom{n}{1}a + \binom{n}{2}a^2 + \binom{n}{3}a^3 + \cdots + \binom{n}{n-1}a^{n-1} + a^n = (????)^n.$$

   (b) Use the formula from part (a), to compute the value given by $\displaystyle\sum_{k=0}^{12} \binom{12}{k}$.

   (c) What is 11^4? How can you use the formula in part (a) to easily compute this value by hand?