Math 310 Assignment 5

PROBLEMS: 4.45, 4.47, 5.8, 5.30

In addition, complete the following additional problems:

1. Using results from class give a short justification that \( \mathbb{R} - \mathbb{Q} \) (the set of irrational numbers) is uncountable? Thus, in the sense of cardinality, the irrational numbers form a larger order of infinity than the rational numbers.

2. Prove that \( \binom{n}{k} = \binom{n}{n-k} \). (Hint: The proof will be short, you can either use a combinatorial argument for why they would give the same number or use the known formula for binomial coefficients.)

3. (a) Using the binomial theorem, tell me what needs to go in place of the question marks in

\[
1 + \binom{n}{1} a + \binom{n}{2} a^2 + \binom{n}{3} a^3 + \cdots + \binom{n}{n-1} a^{n-1} + a^n = (\text{????})^n.
\]

(b) Use the formula from part (a), to compute the value given by \( \sum_{k=0}^{12} \binom{12}{k} \).

(c) What is 11^4? How can you use the formula in part (a) to easily compute this value by hand?

Note: The four questions below are from old midterm exams.

4. Let \( f : A \to A \) be a function. Prove that if \( h = f \circ f \) is a bijection, then \( f \) is a bijection.

5. Give a counterexample to the following statement:
   Every onto function from \( \mathbb{R} \) to \( \mathbb{R} \) is one-to-one.

6. The two sums \( A = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \) and \( B = \sum_{k=0}^{n} \binom{n}{k} w^k (1-w)^{n-k} \) each give a fixed constant for all choices of \( n \geq 1 \) and \( w \). Give the two constant values \( A \) and \( B \).

7. Let \( n \geq 2 \) and \( 2 \leq k \leq n \). Prove that \( \binom{n}{k} = \binom{n-2}{k} + 2 \binom{n-2}{k-1} + \binom{n-2}{k-2} \).
   (Hint: Use Pascal’s formula and induction is not necessary.)

8. Let \( n \geq 1 \).
   
   (a) Explain why the three expressions below are equal:
   \[
   \binom{2n}{n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{1 \cdot 2 \cdot 3 \cdots n} 2^n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} 2^n.
   \]

   (b) Prove that \( 2^n < \binom{2n}{n} < 2^{2n} \).

The problems above are DUE FRIDAY, NOVEMBER 7 at lecture or by 3:00pm at my office.
• PROBLEM 4.45: (try the contrapositive or contradiction for each direction) For one direction, you may want to use the so-called Pigeonhole Principle that states “If \( m \) objects are being placed into \( n \) classes and \( m > n \), then some class must contain two or more objects.” That is, a function between finite sets \( A \) and \( B \) with \( |A| > |B| \) must hit some element of \( B \) at least 2 times (by the Pigeonhole Principle). To show that the equivalence fails for infinite sets, give me an example where the theorem is not true for an infinite set.

• PROBLEM 4.47: Give formal proofs, that is give bijections \( f : \) even natural numbers \( \rightarrow \mathbb{N} \) and \( g : \) odd natural numbers \( \rightarrow \mathbb{N} \).

• The problems below are not due, but I will award at least 1 point of extra credit per challenge problem correctly completed.

CHALLENGE PROBLEMS: 4.48, 5.50, 5.57, 5.62