Math 310 Midterm Solutions:

1. (12=4+4+4 points)

(a) Suppose that $P$ is a true statement, while $R$ and $S$ are false statements. Which of the following statements are true? (Circle the true ones; no justification needed)

(i) $(P \lor R) \land S$
(ii) $S \Rightarrow (P \Rightarrow \neg S)$
(iii) $\neg (R \lor Q) \iff S$

(b) Negate the following statement completely (no negation symbol should be left):

$\forall \epsilon \in R^+, \exists \delta \in R^+$ such that $|x - a| < \epsilon \Rightarrow |f(x) - f(a)| < \delta$.

$\exists \epsilon \in R^+, \forall \delta \in R^+$ such that $|x - a| < \epsilon \land |f(x) - f(a)| \geq \delta$.

[Recall that, as discussed in class, the negation of "$P \Rightarrow Q"$ is "$P$ and not $Q$".]

2. (6 points) What is wrong with the following "proof"?

Claim: $\frac{1}{x+1} < \frac{1}{x}$, for all real numbers $x \in \mathbb{R} - \{0, -1\}$.

"Proof":

$$\frac{1}{x+1} < \frac{1}{x}$$
$$\Rightarrow x < x + 1$$
$$\Rightarrow 0 < 1$$

Since $0 < 1$, it is true that $\frac{1}{x+1} < \frac{1}{x}$ for all real numbers $x \in \mathbb{R} - \{0, -1\}$. QED

There are two major problems with this "proof".

1) The first one is that the argument (chain of implications) is backwards. The author starts with the desired conclusion (namely $\frac{1}{x+1} < \frac{1}{x}$), and ends with something true ($0 < 1$). This says nothing about the truth value of the desired conclusion.

2) The second error is in claiming that $\frac{1}{x+1} < \frac{1}{x} \Rightarrow x < x + 1$ for all reals except 0 and -1. To get $x < x + 1$ from $\frac{1}{x+1} < \frac{1}{x}$, the author implicitly multiplies both sides of $\frac{1}{x+1} < \frac{1}{x}$ by $x(x+1)$ (i.e. by both $x$ and $x+1$). For the inequality direction to be preserved, as claimed, one needs $x(x+1) \geq 0$, i.e. this implication is only valid when $x$ and $x + 1$ have the same sign (which, if you figure it out, only holds for $x \in (-\infty, -1) \cup (0, \infty)$.)
(14=8+6 points)

(a) Circle the statements that are always true for any sets $A$ and $B$ (no proof needed):

1. $A \in P(A)$ (TRUE, since $A \in P(A)$ is equivalent to $A \subseteq A$)
2. $A \subseteq P(A)$ (FALSE. For this to be true, all elements of $A$ would have to also be subsets of $A$)
3. $\{A\} \in P(A)$ (FALSE. By def of $P(A)$, $\{A\} \in P(A)$ is equivalent to $\{A\} \subseteq A$, i.e. $A \in A$)
4. $\{A\} \in P(P(A))$ (TRUE: $\{A\} \in P(P(A)) \iff \{A\} \subseteq P(A) \iff A \in P(A) \iff A \subseteq A$.)
5. $B - A \subseteq B$ (TRUE: $x \in B - A \Rightarrow (x \in B \text{ and } x \notin A) \Rightarrow x \in B$)
6. $A \in A$ (ALWAYS FALSE)
7. $\emptyset \in A$ (FALSE as a universal statement, though there exist some sets for which it is true. For instance, if $A = \{1, 2\}$, then $\emptyset \notin A$. On the other hand, if $A = \{1, 2, \emptyset\}$, then $\emptyset \in A$)
8. $A \cap B \subseteq A \cup B$ (TRUE: if an element $x$ is in both $A$ and $B$, then certainly $x$ is in $A$ or $B$)

(b) Prove that, for any sets $A$, $B$, and $C$, if $A \subseteq B \cup C$ and $A \cap B = \emptyset$, then $A \subseteq C$.

Proof: see the solutions to the review problems for midterm. This is exactly one of the midterm practice problems, just with the set names swapped around.
(10=5+5 points)

(a) Is the function \( f : [0, \infty) \to [0, 1), \) \( f(x) = \frac{x}{x+1} \) injective? Prove your answer.

Yes, the function is injective. To see this, suppose that \( f(x_1) = f(x_2) \) for some \( x_1, x_2 \) in the domain. That is,
\[
\frac{x_1}{x_1+1} = \frac{x_2}{x_2+1}.
\]
Then, cross-multiplying, \( x_1(x_2 + 1) = x_2(x_1 + 1) \). Distributing, \( x_1x_2 + x_1 = x_2x_1 + x_2 \). Subtracting \( x_1x_2 \) from both sides (and by commutativity of multiplication) \( x_1 = x_2 \). QED

(b) Let \( g \) be the function obtained from the function \( f \) in part (a) by restricting its domain to \([0, 1)\) (that is, \( g = f\mid_{[0,1)} \)). Compute \( g \circ f \) and indicate its domain and codomain.

Note that \( g \) was obtained from \( f \) by the restriction of its domain only, so \( g : [0, 1) \to [0, 1), g(x) = f(x) = \frac{x}{x+1} \) for all \( x \in [0, 1) \).

The domain of \( g \circ f \) is the domain of \( f \), and the codomain is the codomain of \( g \) (Ch 8, pg 94-95). That is: \( g \circ f : [0, \infty) \to [0, 1) \).

\[
(g \circ f)(x) = g(f(x)) = \frac{x}{x+1} + 1 = \frac{x}{2x+1} = \frac{x}{2x+1}
\]
(10 points) Answer only one of the following two problems. Don’t include your scratch work, and write your proof clearly and carefully.

5. Prove that \( \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \), for all integers \( n \geq 1 \).

This is very similar to a problem in the book which was done in class, and to one of the midterm practice problem. Feel free to ask me in person if any clarification is needed.

OR

6. Prove by contradiction that if two integer numbers \( m \) and \( n \) have no common factor, then \( m+n \) and \( m \) have no common factor either.

(Definition: Two integers \( a \) and \( b \) have a common factor \( q \) iff \( q \) divides both of them: i.e. \( a = qa' \) and \( b = qb' \) for some integers \( a', b' \). Here \( q \) needs to be an integer and \( q \neq \pm 1 \).)

PROOF:
Suppose, by contradiction, that there exists two integers \( m \) and \( n \) such that \( m \) and \( n \) do not have a common factor, but \( m+n \) and \( m \) do have a common prime factor \( q \). That is \( m = qa \) and \( m+n = qb \) for some integers \( a \) and \( b \). Then, solving for \( n \) and then replacing \( m = qa \):

\[ n = qb - m = qb - qa = q(b - a) \]

Since \( b - a \) is an integer, \( q \) divides \( n \). But \( q \) also divides \( m \), so we showed that \( m \) and \( n \) have a common factor \( q \). This contradicts the hypothesis that \( m \) and \( n \) do not have a common factor. QED

Note: What we really proved is the contrapositive of the desired implication.