Math 307 E - Summer 2011
Practice Midterm 2
August 17, 2011

Name: ______________________ Student number: ______________

1	10	
2	10	
3	10	
4	10	
5	10	
6	3*	
Total	50	

- Complete all questions.
- You may use a scientific calculator during this examination. Other electronic devices (e.g. cell phones) are not allowed, and should be turned off for the duration of the exam.
- You may use one hand-written 8.5 by 11 inch page of notes.
- Show all work for full credit.
- You have 60+ minutes to complete the exam.

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1. Find the general solution to the differential equations:

   (a) (5 points)

   $$y'' - 2y' - 3y = te^t.$$

   (b) (5 points)

   $$y'' - 2y' - 3y = g(t)$$

   Hint: Express your answer using integrals.

2. (10 points) Suppose that the motion of a spring-mass system satisfies

$$u'' + u' + 1.5u = \sin(2t)$$

and that the mass starts $(t = 0)$ at the equilibrium position from rest. Find the the position $u(t)$ at any time t.

3. Compute the following Laplace transforms using the definition, or using only the numbers 1,13,14,18, and 19 on the table.

(a) (5 points)

$$\mathcal{L}\left\{t^2 e^{\pi t}\right\}$$

(b) (5 points)

$$\mathcal{L}\left\{u_3(t)(t^2 - 2t - 1)\right\}$$

4. (10 points) Use the Laplace transform to solve the following IVP using the table:

$$y'' - y = \begin{cases} 1 & t < 2 \\ t/3 & 2 \leq t \end{cases} \qquad \begin{cases} y(0) = 0 \\ y'(0) = 0. \end{cases}$$

5. (10 points) A spring-mass system has a spring constant of 2N/m. A mass of 8kg is attached to the spring. Let γ be the damping constant of the system.

   (a) (2 points) What is the *natural frequency* of the system?

   (b) (2 points) Suppose $\gamma = 9$. Is the (free) system under-damped, over-damped or critically damped?

   (c) (2 points) From now on, suppose $\gamma = 2$. Find the quasi-frequency of the (free) system.

   (d) (2 points) Suppose we apply an external force $F(t) = 5\cos(wt)$ N. What is the resonant frequency of this forced system?

   (e) (2 points) Write down the initial value problem corresponding to this forced system where w is the resonant frequency, and the mass starts at rest from the equilibrium position.

6. (3 bonus points) Compute the laplace transform of $\ln(t)$ by following these steps.

(a) (1 point) Differentiate the formula

$$\mathcal{L}(t^p) = \int_0^\infty e^{-st} t^p dt = \frac{\Gamma(p+1)}{s^{p+1}}$$

with respect to p. For the the middle term, move the differential operator $\frac{d}{dp}$ inside the integral and apply it to the integrand.

(b) (1 point) Simplify as much as possible, and then evaluate the resulting expression at $p = 0$.

(c) (1 point) What is $\mathcal{L}(\ln(t))$?

Table of Laplace transforms:

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}, \quad s > 0$
2. e^{at}	$\frac{1}{s-a}, \quad s > a$
3. $t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4. $t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
6. $\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
7. $\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
8. $\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
9. $e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10. $e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11. $t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}$
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t) f(t-c)$	$e^{-cs} F(s)$
14. $e^{ct} f(t)$	$F(s-c)$
15. $f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right), c > 0$
16. $\int_0^t f(t-\tau) g(\tau) d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}
18. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
19. $(-t)^n f(t)$	$F^{(n)}(s)$