1. (10 pts) Find the explicit solution to $y' = 10e^y x\sin(x^2)$ with $y(0) = 0$.

SEPARABLE!

$$\int e^{-y}\,dy = \int 10x\sin(x^2)\,dx$$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2x}du = dx$

$$-e^{-y} = \int 5\sin(u)\,du$$

$$-e^{-y} = -5\cos(u) + C_1$$

$$e^{-y} = 5\cos(x^2) + C_2$$

$C_2 = -C_1$

$$-y = \ln(5\cos(x^2) + C_2)$$

$$y = -\ln(5\cos(x^2) + C_2)$$

$y(0) = 0 \Rightarrow 0 = -\ln(5 + C_2) \Rightarrow 5 + C_2 = 1 \Rightarrow C_2 = -4$

$$\boxed{y = -\ln(5\cos(x^2) - 4)}$$

2. (10 pts) One solution to $t^2y'' - 6y = 0$ is $y_1(t) = t^3$. Use reduction of order to find the general explicit solution.

$$y = u(t)\,t^3 \Rightarrow y' = u't^3 + 3ut^2$$

$$y'' = u''t^3 + \underbrace{3u't^2 + 3u't^2}_{6u't^2} + 6ut$$

$$t^2y'' - 6y = 0$$

$$\Rightarrow t^2\left(u''t^3 + 6u't^2 + 6ut\right) - 6\left(ut^3\right) \stackrel{?}{=} 0$$

$$\Rightarrow u''t^5 + 6u't^4 + \cancel{6ut^3} - \cancel{6ut^3} \stackrel{?}{=} 0$$

$v = u'$

$$\Rightarrow v't^5 + 6vt^4 = 0 \Rightarrow \frac{dv}{dt}t^5 = -6vt^4$$

$$\Rightarrow \frac{dv}{dt} = \frac{-6v}{t}$$

$$\Rightarrow \frac{1}{v}dv = -\frac{6}{t}dt$$

$$\ln|v| = -6\ln|t| + C_1 = \ln|t^{-6}| + C_1$$

$$\Rightarrow |v| = e^{(\ln(t^{-6}) + C_1)}$$

$$\Rightarrow v = \underbrace{\pm e^{C_1}}_{C_2} e^{\ln(t^{-6})}$$

$$v = C_2 t^{-6} = u'$$

$$\Rightarrow u(t) = \frac{C_2}{-5}t^{-5} + D$$

$$\Rightarrow u(t) = Ct^{-5} + D$$

$\boxed{\text{GENERAL SOL'N}}$

$$\boxed{y = (Ct^{-5} + D)\,t^3 = Ct^{-2} + Dt^3}$$

3. (15 pts) The two parts below are NOT related.

(a) A home buyer gets a 30-year home loan for \$300,000 at 5% annual interest compounded continuously. Assume the buyer pays K dollars each month (so $12K$ dollars each year), where K is a constant. Let $y(t)$ be the balance of the loan after t years. As we did in class and in homework, model this situation by assuming the rate of change of the balance is equal to the yearly interest minus the yearly payments (where yearly interest is proportional to the balance with proportionality constant $r = 0.05$).
Give the differential equation and the two initial conditions. DO NOT SOLVE, just set up.

$$\boxed{\frac{dy}{dt} = \underbrace{0.05y}_{\text{interest add to balance per year}} - \underbrace{12K}_{\text{payment subtracted from balance per year}}, \quad y(0) = 300{,}000, \quad y(30) = 0}$$

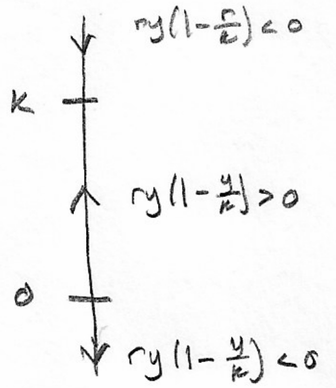
$y(0) = 300{,}000$ ← STARTING BALANCE

$y(30) = 0$ ← ENDING BALANCE

(b) The Logistic equation is the differential equation $\frac{dy}{dt} = ry\left(1 - \frac{1}{K}y\right)$, where r and K are positive constants. Solutions to this equation have been used to accurately model the size of some populations. (Your answers below might involved r and/or K)

i. Find and classify all equilibrium solutions for the Logistic equation as stable, semistable, or unstable.

$ry\left(1 - \frac{y}{K}\right) = 0$ WHEN $y = 0$ OR $y = K$



$$\boxed{\begin{array}{l} y(t) = 0 \text{ IS UNSTABLE} \\ y(t) = K \text{ IS STABLE} \end{array}}$$

ii. If $y = y(t)$ is the solution to the Logistic equation that also satisfies $y(0) = y_0$ with $0 < y_0 < K$, then what is $\lim_{t\to\infty} y(t)$?

Since $\frac{dy}{dt}$ is POSITIVE FOR ALL y BETWEEN 0 & K

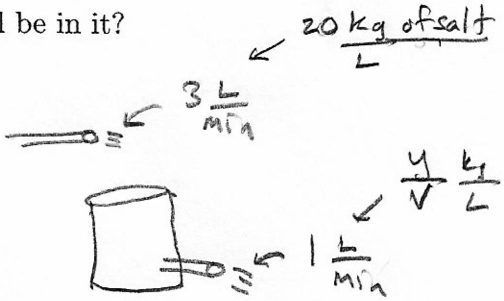
$$\boxed{\lim_{t\to\infty} y(t) = K}$$

4. (12 pts) A 16 liter vat initially contains 1 liter of pure water (no salt). Salt water containing 20 grams/liter of salt enters the vat at 3 liters/minute. The vat is well mixed and the mixture leaves the vat through a hole in the bottom at a constant rate of 1 liter/minute.
Let $y(t)$ be the amount of salt in the vat at time t minutes.
The differential equation is $\frac{dy}{dt} = A - \frac{y}{1+2t}$ with $y(0) = B$, where A and B are constants you should know from the description.

(a) From the description, what are the values of A and B?

$$\boxed{\begin{aligned} A &= 20 \cdot 3 = 60 \ \tfrac{kg}{min} \\ B &= y(0) = 0 \ kg \end{aligned}}$$

(b) At the instant when the vat becomes full, how much salt will be in it? (First, solve the *linear* differential equation).



volume is increasing at $(3-1) = 2 \ \frac{L}{min}$

$V(t) = 1 + 2t$

FULL $\Rightarrow V(t) = 16$
$\Rightarrow 1 + 2t = 16$
$2t = 15$
$t = 7.5$

$$\frac{dy}{dt} + \frac{1}{1+2t} y = 60$$

$$\mu(t) = e^{\int \frac{1}{1+2t} dt} = e^{\frac{1}{2}\ln(1+2t)} = e^{\ln(\sqrt{1+2t})}$$

$$= \sqrt{1+2t}$$

$$\Rightarrow \sqrt{1+2t}\frac{dy}{dt} + \frac{\sqrt{1+2t}}{1+2t} y = 60\sqrt{1+2t}$$

$$\frac{d}{dt}\left(\sqrt{1+2t}\, y\right) = 60(1+2t)^{1/2}$$

$$\sqrt{1+2t}\, y = \int 60(1+2t)^{1/2} dt$$

$$\sqrt{1+2t}\, y = \frac{60}{2}\frac{2}{3}(1+2t)^{3/2} + C$$

$$y = \frac{20(1+2t)^{3/2}}{\sqrt{1+2t}} + \frac{C}{\sqrt{1+2t}} = 20(1+2t) + \frac{C}{\sqrt{1+2t}}$$

$$y(0) = 0 \Rightarrow 0 = 20 + C \Rightarrow C = -20$$

$$y(7.5) = 20(1 + 2\cdot(7.5)) - \frac{20}{\sqrt{1+2\cdot 7.5}} = 20 \cdot 16 - \frac{20}{4}$$

$$= \boxed{315 \ kg}$$

5. (17 pts) **For all parts below**, the mass-spring system satisfies $u'' + \gamma u' + 4u = 0$, where γ is the damping constant. Distances are in inches and time is in seconds.

(a) Assume there is no damping and $u(0) = 1$ inches and $u'(0) = 6$ inches/second. Give the largest value of $u(t)$. (Hint: First solve; your answer will be in inches)

$\gamma = 0 \Rightarrow r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$u = c_1 \cos(2t) + c_2 \sin(2t)$ $\qquad$ $y(0) = 1 \Rightarrow c_1 = 1$

$u' = -2c_1 \sin(2t) + 2c_2 \cos(2t)$ $\qquad$ $y'(0) = 6 \Rightarrow 2c_2 = 6 \Rightarrow c_2 = 3$

$u = \cos(2t) + 3\sin(2t)$ $\qquad$ AMPLITUDE $= \sqrt{1^2 + 3^2} = \boxed{\sqrt{10} \text{ inches}}$

(b) Assume there is no damping and the mass is set in motion. How long does it take for the mass to go from its lowest point back to its lowest point again?

$\omega_0 = \sqrt{\frac{k}{m}} = 2 \ \frac{\text{RAD}}{\text{SEC}}$

PERIOD $= \frac{2\pi}{2} = \boxed{\pi \text{ seconds}}$

(c) If the system is critically damped, give the general explicit solution.

$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 16}}{2}$ ← WANT THIS ZERO

$\gamma^2 - 16 = 0 \Rightarrow \gamma^2 = 16 \Rightarrow \boxed{\gamma = 4}$ $\frac{\text{lbs}}{\text{in/sec}}$

(d) If $\gamma = \sqrt{7}$ lbs/(in/sec), give the general explicit solution.

$r^2 + \sqrt{7} r + 4 = 0$ $\qquad$ $r_{1,2} = \frac{-\sqrt{7} \pm \sqrt{7 - 16}}{2} = \frac{-\sqrt{7} \pm 3i}{2}$

$= -\frac{\sqrt{7}}{2} \pm \frac{3}{2} i$ $\qquad$ $\lambda = -\sqrt{7}/2, \ \mu = 3/2$

$$\boxed{y = e^{-\frac{\sqrt{7}}{2} t} \left(c_1 \cos\left(\tfrac{3}{2} t\right) + c_2 \sin\left(\tfrac{3}{2} t\right) \right)}$$

6. (12 pts) Give the general explicit solution to $y'' + y' - 2y = 4t - 30\sin(t)$.

$r^2 + r - 2 = 0 \Rightarrow (r+2)(r-1) = 0 \Rightarrow r_1 = -2,\ r_2 = 1$

HOMOGENEOUS SOL'NS: $y_1 = e^{-2t}$, $y_2 = e^t$

$y_p(t) = (At + B + C\cos(t) + D\sin(t)) \cdot -2$

$y_p'(t) = (A - C\sin(t) + D\cos(t)) \cdot 1$

$y_p''(t) = (-C\cos(t) - D\sin(t)) \cdot 1$

$$\Rightarrow \underbrace{-2A}_{4}t + \underbrace{(-2B + A)}_{0} + \underbrace{(-C + D - 2C)}_{0}\cos(t) + \overbrace{(-D - C - 2D)}^{-30}\sin(t) \stackrel{?}{=} 4t - 30\sin(t)$$

$\Rightarrow -2A = 4 \Rightarrow A = -2$

$-2B + A = 0 \Rightarrow -2B - 2 = 0 \Rightarrow B = -1$

$D - 3C = 0 \Rightarrow D = 3C$

$-3D - C = -30 \Rightarrow -9C - C = -30 \Rightarrow C = 3 \Rightarrow D = 9$

$$\boxed{y = c_1 e^{-2t} + c_2 e^t - 2t - 1 + 3\cos(t) + 9\sin(t)}$$

7. (12 pts)

(a) Give the Laplace transformation of $f(t) = t^3 + e^{-7t} + (t+4)u_6(t)$

$$\frac{3!}{s^4} + \frac{1}{s+7} + e^{-6t}\mathcal{L}\{(t+6)+4\}$$

$$\boxed{\frac{6}{s^4} + \frac{1}{s+7} + e^{-6t}\left(\frac{1}{s^2} + \frac{10}{s}\right)}$$

(b) Give the inverse Laplace transformation of $F(s) = \dfrac{3s+2}{(s-1)^2+4} + \dfrac{e^{-2s}}{s-5}$

$$\frac{3s+2}{(s-1)^2+4} = \frac{A(s-1)+B}{(s-1)^2+4} \Rightarrow As - A + B = 3s+2$$

$$A = 3 \Rightarrow -3 + B = 2 \Rightarrow B = 5$$

$$\frac{3(s-1)}{(s-1)^2+4} + \frac{5}{(s-1)^2+4} + e^{-2s}\frac{1}{s-5}$$

$$3e^t\cos(2t) + \frac{5}{2}e^t\sin(2t) + u_2(t)\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\}}_{e^{5t}}\Big|_{t\to t-2}$$

$$\boxed{3e^t\cos(2t) + \frac{5}{2}e^t\sin(2t) + u_2(t)e^{5(t-2)}}$$

8. (12 pts) You make a cup of tea with an initial temperature of 200°F, which is too hot. You place it in a 30°F freezer for 0.2 hours (12 minutes), then take it out and set it on a table where room temperature is 70°F. Assume Newton's law of cooling with a cooling constant of $k = 1$, we get

$$\frac{dy}{dt} = -(y - y_s(t)), \text{ where } y_s(t) = \begin{cases} 30 & , 0 \le t < 0.2; \\ 70 & , t \ge 0.2, \end{cases}$$

where $y = y(t)$ is the temperature of the tea after t hours.
Using Laplace transforms, solve for the function $y(t)$ **and** give the temperature of the tea 20 minutes ($t = 1/3$ hours) after you made it, to the nearest degree.

$$y' + y = y_s(t) = 30 + 40u_{0.2}(t)$$

$$s\mathcal{L}\{y\} - y(0) + \mathcal{L}\{y\} = \frac{30}{s} + e^{-0.2t}\frac{40}{s}$$

$$(s+1)\mathcal{L}\{y\} = 200 + \frac{30}{s} + e^{-0.2t}\frac{40}{s}$$

$$\mathcal{L}\{y\} = \frac{200}{s+1} + 30\frac{1}{s(s+1)} + 40e^{-0.2t}\frac{1}{s(s+1)}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \Rightarrow \begin{matrix} A = 1 \\ B = -1 \end{matrix} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{-1}{s+1}\right\} = 1 - e^{-t}$$

$$y = 200e^{-t} + 30(1 - e^{-t}) + 40u_{0.2}(t)\left(1 - e^{-t}\Big|_{t \to t-0.2}\right)$$

$$y(t) = 30 + 170e^{-t} + 40u_{0.2}(t)\left(1 - e^{-(t-0.2)}\right)$$

$$= \begin{cases} 30 + 170e^{-t} & , 0 \le t < 0.2; \\ 70 + 170e^{-t} - 40e^{-(t-0.2)} & , t \ge 0.2. \end{cases}$$

$$y(0.\overline{3}) = 70 + 170e^{-1/3} - 40e^{-(\frac{1}{3} - 0.2)} \approx 156.8$$

$$\approx \boxed{157°F}$$

ASIDE: IF WAS 169.18°F at $t = 0.2$