Math 307 I - Spring 2011
Practice Final
June 03, 2011

Name: ______________________ Student number: ______________

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	1	
Total	91	

- Complete all questions.
- You may use a scientific calculator during this examination. Other electronic devices (e.g. cell phones) are not allowed, and should be turned off for the duration of the exam.
- You may use one hand-written 8.5 by 11 inch page of notes.
- Show all work for full credit.
- You have 120 minutes to complete the exam.

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1. Find the general solution to the differential equations:

   (a) (5 points)

$$y' = (2 - e^x)/(3 + 2y)$$

This equation is separable: $(3 + 2y)dy = (2 - e^x)dx$. We integrate and get

$$3y + y^2 = 2x - e^x + C.$$

This is enough, but if you want to solve for y you can use the quadratic formula:

$$y = \frac{-3}{2} \pm \frac{1}{2}\sqrt{9 + 4(2x - e^x + C)}.$$

   (b) (5 points)

$$ty' + 2y = (\sin\ t)/t.$$

This is a linear (non-separable) equation, so we know we're going to use integrating factors. We must first normalize the equation by dividing by t:

$$y' + \frac{2}{t}y = \frac{\sin t}{t^2}.$$

By the method of integrating factors, we multiply by $\mu(t) = e^{\int \frac{2}{t}dt} = t^2$, and get

$$t^2y' + 2ty = \sin t.$$

Rewriting, we have $\frac{d}{dt}(t^2y) = \sin(t)$; this can be solved by integrating both sides. We get $t^2y = -\cos(t) + C$, or

$$y = \frac{-1}{t^2}\cos(t) + \frac{C}{t^2}.$$

2. Find the general solution to the differential equations:

   (a) (5 points)

$$y' = \frac{y^3 - x^3}{2x^3}$$

This simplifies as $y' = \frac{1}{2}\left(\frac{y}{x}\right)^3 - \frac{1}{2}$. This is not linear or separable; but the right hand side is homogeneous, so we try the substitution $u = \frac{y}{x}$, or $y = ux$. Then $\frac{dy}{dx} = u + x\frac{du}{dx}$, so the differential equation becomes

$$u + x\frac{du}{dx} = \frac{1}{2}u^3 - \frac{1}{2}.$$

We subtract u from both sides — now it is separable! We get

$$\frac{2\,du}{u^3 - 2u - 1} = \frac{dx}{x}.$$

To integrate the LHS, we factor

$$u^3 - 2u - 1 = (u+1)(u^2 - u - 1) = (u+1)(u - \frac{1+\sqrt{5}}{2})(u - \frac{1-\sqrt{5}}{2})$$

and use partial fractions to write

$$\frac{2}{(u+1)(u - \frac{1+\sqrt{5}}{2})(u - \frac{1-\sqrt{5}}{2})} = \frac{2}{u+1} + \frac{2/(5+3\sqrt{5}/2)}{u - \frac{1+\sqrt{5}}{2}} + \frac{4/(3-\sqrt{5})}{u - \frac{1-\sqrt{5}}{2}}$$

Since these numbers are so ugly, it is easier to just write the LHS as

$$\frac{2}{u+1} + \frac{A}{u - r_1} + \frac{B}{u - r_2}.$$

Then integrating both sides, we get:

$$2\ln(u+1) + A\ln(u - r_1) + B\ln(u - r_2) = \ln(x) + C,$$

and exponentiating both sides we get:

$$(u+1)^2(u - r_1)^A(u - r_2)^B = C_2 x \qquad \text{or} \qquad (y/x + 1)^2(y/x - r_1)^A(y/x - r_2)^B = C_2 x.$$

This is the solution with A, B, r_1, r_2 as indicated.

(b) (5 points)

$$\cos(y)dt + (y^2 - t\sin(y))dy = 0.$$

This equation is not separable or linear, and is not homogeneous. So we test for exactness:

$$\frac{\partial}{\partial y}(\cos y) = -\sin(y) = \frac{\partial}{\partial t}(y^2 - t\sin y).$$

So the equation is exact! To solve it, we must find the potential function $\phi(t, y)$. We know that

$$\frac{\partial}{\partial t}\phi = \cos(y),$$

so that $\phi(t, y) = t\cos(y) + g(y)$, where $g(y)$ is a function depending only on y (and not on t). We also know that

$$\frac{\partial}{\partial y}\phi = \frac{\partial}{\partial y}(t\cos(y) + g(y)) = y^2 - t\sin(y),$$

so that $-t\sin(y) + g'(y) = y^2 - t\sin(y)$. This shows that $g'(y) = y^2$, so that $g(y) = \frac{1}{3}y^3 + C$. Any constant will do, so we choose $C = 0$.

Finally, our differential equation can be re-written as $\frac{d}{dt}\phi(t, y) = 0$, so after integrating the solution is

$$\phi(t, y) = t\cos(y) + \frac{1}{3}y^3 = C_2.$$

3. (10 points) Bill is twenty-five years from retirement; in order to retire, Bill needs $500,000 in his saving's account when he retires in order to maintain his current standard of living. If Bill has $100,000 in his saving's account right now, and the account earns 5% annually (compounded continuously), how much does Bill need to save each year to reach his goal? (Assume that Bill *continuously* deposits this annual sum into his saving's account.)

We measure time in years. If $y(t)$ denotes the amount of money in Bill's saving's account t year's from now, then we model $y(t)$ by:

$$\frac{dy}{dt} = ry + k,$$

where r is the interest rate and k is the amount deposited each year (deposited continuously). So we have:

$$y' = 0.05y + k.$$

This is a separable equation: $\frac{dy}{0.05y+k} = dt$, and integrating yields $20\ln(0.05y + k) = t + C$. Solving for y:

$$y = C_2 e^{t/20} - 20k.$$

We may solve for C_2, since we know that $y(0) = 100,000$. Then $100,000 + 20k = C_2$. Finally, we solve for k using that $y(25) = 500,000$. Then

$$\begin{aligned} & 500,000 = (100,000 + 20k)e^{25/20} - 20k. \\ \Rightarrow \quad & 500,000 - 100,000e^{25/20} = 20k(e^{25/20} - 1) \\ \Rightarrow \quad & \frac{500,000 - 100,000e^{25/20}}{20(e^{25/20} - 1)} = k = \$3031.02. \end{aligned}$$

4. Solve the following IVP's:

(a) (5 points)

$$y'' - 3y' + 4y = 0 \qquad \begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$$

The characteristic equation is $r^2 - 3r + 4 = 0$, so using the quadratic formula we have $r = \frac{3 \pm \sqrt{9-4(4)}}{2} = \frac{3}{2} \pm \frac{\sqrt{7}}{2}i$. Then we can choose either root, say $r = \frac{3}{2} + \frac{\sqrt{7}}{2}i$, and find the real and imaginary parts of

$$e^{rt} = e^{3t/2}\left[\cos\left(\frac{\sqrt{7}}{2}t\right) + i\sin\left(\frac{\sqrt{7}}{2}t\right)\right].$$

(Here we have used Euler's formula, as usual.) Then the general solution is

$$y(t) = c_1 e^{3t/2}\cos(t\sqrt{7}/2) + c_2 e^{3t/2}\sin(t\sqrt{7}/2).$$

(b) (5 points)

$$y'' - 3y' + 4y = \sin(t) \qquad \begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$$

We have the same homogeneous solution as the previous case, so we need to find a particular solution. I will use the method of undetermined coefficients: let $y_p(t) = A\sin t + B\cos t$. Then

$$\begin{aligned} \sin(t) &= y_p'' - 3y_p' + 4y_p \\ &= [-A\sin t - B\cos t] - 3[A\cos t - B\sin t] + 4[A\sin t + B\cos t] \\ &= [-A + 3B + 4A]\sin t + [-B - 3A + 4B]\cos t. \end{aligned}$$

We have the system of equations $3A + 3B = 1$ and $-3A + 3B = 0$. Thus $A = B = \frac{1}{6}$. So

$$y(t) = y_h(t) + y_p(t) = c_1 e^{3t/2}\cos(t\sqrt{7}/2) + c_2 e^{3t/2}\sin(t\sqrt{7}/2) + \frac{1}{6}(\sin t + \cos t).$$

Now we use the initial conditions to find c_1 and c_2:

$$\begin{aligned} y(0) &= 1 = c_1 + \frac{1}{6} \\ y'(0) &= 0 = \frac{3}{2}c_1 + \frac{\sqrt{7}}{2}c_2 + \frac{1}{6}. \end{aligned}$$

Solving this system gives $c_1 = \frac{5}{6}$, and $c_2 = -\frac{17}{6\sqrt{7}}$. Thus

$$y(t) = \frac{5}{6}e^{3t/2}\cos(t\sqrt{7}/2) + -\frac{17}{6\sqrt{7}}e^{3t/2}\sin(t\sqrt{7}/2) + \frac{1}{6}(\sin t + \cos t)$$

5. Find the general solution to the following differential equations:

(a) (5 points)

$$y'' - 4y' + 4y = 0$$

The characteristic equation is $r^2 - 4r + 4 = (r-2)^2$; since $r = 2$ is a double root, the general solution is

$$y = c_1 e^{2t} + c_2 t e^{2t}.$$

(b) (5 points)

$$y'' - 3y' + 2y = e^{2t}.$$

The characteristic equation is $r^2 - 3r + 2 = (r-2)(r-1)$. So the homogeneous solution is $y_h(t) = c_1 e^{2t} + c_2 e^t$. The particular solution is of the form $y_p(t) = Ate^t$. (We must multiply by t^1 because 2 is a root of the homogeneous equation of order 1.) Then we solve for A:

$$\begin{aligned} e^{2t} &= y_p'' - 3y_p' + 2y_p \\ &= [2Ae^t + Ate^t] - 3[Ae^t + Ate^t] + 2Ate^{2t} \\ &= Ae^t[2 + t - 3 - 3t + 2t] = -Ae^2. \end{aligned}$$

We conclude that $A = -1$, so that $y_p(t) = -te^t$. Finally,

$$y(t) = y_h(t) + y_p(t) = c_1 e^{2t} + c_2 e^t - te^t.$$

6. (10 points) Suppose that the motion of a spring-mass system satisfies

$$u'' + u' + 1.5u = \sin(t)$$

and that the mass starts $(t = 0)$ at the equilibrium position from rest. Find the *steady-state solution* (the approximate solution for large values of t).

The characteristic equation is $r^2 + r + 1.5 = 0$, so $r = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5}i$. It follows from Euler's equation that the homogeneous solution is

$$y_h(t) = e^{-t/2}(c_1 \cos(t\sqrt{5}/2) + c_2 \sin(t\sqrt{5}/2)).$$

As $t \to \infty$, this goes to zero, so this is part of the *transient solution*, and not the steady-state solution.

Now we find the particular solution. It must be of the form $u_p(t) = A\cos(t) + B\sin(t)$. Alternatively, we can replace the driving function $\sin(t)$ with e^{it}, since $Im(e^{it}) = \sin(t)$. We take this second approach here. Then we let $y_p(t) = Ae^{it}$, and

$$\begin{aligned} e^{it} &= y_p'' + y_p' + 1.5y_p \\ &= [-Ae^{it}] + [Aie^{it}] + 1.5[Ae^{it}] \\ &= Ae^{it}[-1 + i + 1.5] = (1/2 + i)Ae^{it}. \end{aligned}$$

It follows that $A = \frac{1}{1/2+i} = \frac{2}{5} - \frac{4}{5}i$. Finally, $y_p(t) = (2/5 - 4/5i)e^{it} = [2/5\cos(t) + 4/5\sin 9t)] + i[2/5\sin(t) - 4/5\cos(t)]$.

Since we are interested in the imaginary part of the solution, we get that

$$u_p(t) = Im(y_p(t)) = 2/5\sin(t) - 4/5\cos(t).$$

This is the steady-state solution.

7. (10 points) Compute the following Laplace transform using the definition (i.e. without using the table):

$$\mathcal{L}\left\{t\, e^{at}\right\}$$

By definition:

$$\begin{aligned}
\mathcal{L}\left\{t\, e^{at}\right\} &= \int_0^\infty e^{-st} \cdot te^{at} dt = \int_0^\infty e^{-(s-a)t} t dt \\
&= \frac{te^{-(s-a)t}}{-(s-a)}\Big|_{t=0}^{t=\infty} + \frac{1}{s-a}\int_0^\infty e^{-(s-a)t} dt \\
&= 0 + \frac{-e^{-(s-a)t}}{(s-a)^2}\Big|_{t=0}^{t=\infty} = 0 + \frac{1}{(s-a)^2}.
\end{aligned}$$

For the second line, we did integration by parts.

8. (10 points) Find the inverse Laplace transform of

$$F(s) = \frac{e^{-\pi s} - e^{-2\pi s}}{s(s-1)(s-2)}$$

using the table.

The inverse Laplace transform is linear, so we can split up the problem as

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{e^{-\pi s}H(s)\right\} - \mathcal{L}^{-1}\left\{e^{-2\pi s}H(s)\right\},$$

where $H(s) = \frac{1}{s(s-1)(s-2)}$. Each of the functions on the right is in the form where we can use the *translation formula*: number 13 on the Laplace transform sheet. However, in order to use this formula we need to know $h(t) = \mathcal{L}^{-1}\{H(s)\}$. Since $H(s)$ is a rational function (a quotient of polynomials), we need to use partial fractions to put it in a form we can identify on the table. Use the cover-up method to get

$$H(s) = \frac{1}{s(s-1)(s-2)} = \frac{1/2}{s} + \frac{-1}{s-1} + \frac{1/2}{s-2};$$

then we have that

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\{H(s)\} \\ &= 1/2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 1/2\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} \\ &= 1/2 \cdot 1 - 1 \cdot e^{t} + 1/2 \cdot e^{2t}. \end{aligned}$$

Now that we know $h(t)$, we can use the translation formula to get

$$\begin{aligned} \mathcal{L}^{-1}\left\{e^{-\pi s}H(s)\right\} &= u_\pi(t)h(t-\pi) = u_\pi(t)\left(1/2 - e^{t-\pi} + 1/2e^{2(t-\pi)}\right) \\ \mathcal{L}^{-1}\left\{e^{-2\pi s}H(s)\right\} &= u_{2\pi}(t)h(t-2\pi) = u_{2\pi}(t)\left(1/2 - e^{t-2\pi} + 1/2e^{2(t-2\pi)}\right). \end{aligned}$$

Finally,

$$\mathcal{L}^{-1}\{F(s)\} = u_\pi(t)\left(1/2 - e^{t-\pi} + 1/2e^{2(t-\pi)}\right) - u_{2\pi}(t)\left(1/2 - e^{t-2\pi} + 1/2e^{2(t-2\pi)}\right).$$

9. (10 points) Use the Laplace transform to solve the following IVP:

$$y'' + y = \begin{cases} t/2, & 0 \le t < 6 \\ t-3, & 6 \le t \end{cases} \qquad \begin{cases} y(0) = 0 \\ y'(0) = 1. \end{cases}$$

We want to take the Laplace transform of both sides, but first we need to write the driving function in a more usable form. Observe that the driving function $g(t) = t/2 + u_6(t)(-t/2 + t - 3)$. This can be re-written as

$$g(t) = t/2 + \frac{1}{2}u_6(t)(t-6).$$

We denote $Y(s) = \mathcal{L}\{y(t)\}$. We take the Laplace transform of both sides and get

$$s^2Y(s) - 1 + Y(s) = \frac{1 + e^{-6s}}{2s^2},$$

so that

$$Y(s) = \frac{1}{s^2+1} + \frac{1}{2}\frac{1+e^{-6s}}{s^2(s^2+1)}.$$

To finish the problem we just need to take the inverse transform of both sides. The first term on the RHS has inverse transform $\sin(t)$. The other term can be broken up as

$$\frac{1}{2}\frac{1+e^{-6s}}{s^2(s^2+1)} = \frac{1}{2}F(s) + \frac{1}{2}e^{-6s}F(s),$$

where $F(s) = \frac{1}{s^2(s^2+1)}$. To evaluate the inverse transform of either term, we need to know $f(t) = \mathcal{L}^{-1}(F(s))$. So we use partial fractions to get

$$F(s) = \frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}.$$

If you "clear the denominators" – you will get

$$1 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2.$$

Now, choose special values of s: namely $s = 0$ and $s = i = \sqrt{-1}$. This gives you that $B = 1$ and then $C = 0, D = -1$. Finally, choose any other value (say $s = 1$) and get $A = 0$. Thus

$$F(s) = \frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1},$$

so that $f(t) = t - \sin(t)$. Then

$$\mathcal{L}^{-1}(e^{-6s}F(s)) = u_6(t)f(t-6) = u_6(t)(t - 6 - \sin(t-6)),$$

so that

$$y(t) = \frac{1}{2}t + \frac{1}{2}\sin(t) + \frac{u_6(t)}{2}(t - 6 - \sin(t-6)).$$

10. (1+ points) Your friend Tim is bad at calculus. He saw you working on the following integral:

$$\int_0^t e^s \cos(s) ds,$$

and suggested that it is equal to

$$\int_0^t e^s ds \cdot \int_0^t \cos(s) ds = (e^t - 1)\sin(t).$$

(a) (1 point) Verify directly that this "solution" is incorrect by computing it's derivative. (Explain what the derivative would be if Tim were correct.)

We differentiate and get $e^t \sin(t) + e^t \cos(t) - \cos(t)$. If Tim were correct, then by the fundamental theorem of calculus, this should be the same function as the integrand, $e^t \cos(t)$, which is not the case. So Tim is incorrect.

(b) (2 bonus points) Tim suggests that he was just "unlucky" with $\cos(s)$, and that his rule "usually works." You are not convinced, so you consider Tim's identity:

$$\int_0^t e^s f(s) ds = (e^t - 1) \int_0^t f(s) ds.$$

By differentiating the equation (both sides) twice – and using the Fundamental theorem of calculus – find a separable differential equation that $f(t)$ must satisfy for Tim's rule to work. Solve it for $f(t)$. Does Tim's rule "usually work"?

We differentiate both sides twice:

$$e^t f(t) = e^t \int_0^t f(s) ds + (e^t - 1) f(t)$$

$$e^t f(t) + e^t f'(t) = e^t f(t) + e^t \int_0^t f(s) ds + e^t f(t) + (e^t - 1) f'(t).$$

The two equations reduce to

$$f(t) = e^t \int_0^t f(s) ds$$

$$f'(t) = e^t f(t) + e^t \int_0^t f(s) ds.$$

Combining these, we get $f'(t) = e^t f(t) + f(t) = (e^t + 1) f(t)$. This is a separable differential equation, which we solve to get $\ln(f) = e^t + t + C$, or

$$f(t) = A e^{e^t} e^t.$$

So Tim's rule does not "usually" work. (In fact, it only works if $f(t) = 0$. But if you replace 0 in the lower limit by $-\infty$, then it will work for precisely the functions found above.)

Table of Laplace transforms:

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}, \quad s > 0$
2. e^{at}	$\frac{1}{s-a}, \quad s > a$
3. $t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4. $t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
6. $\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
7. $\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \|a\|$
8. $\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \|a\|$
9. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11. $t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}$
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct}f(t)$	$F(s-c)$
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), c > 0$
16. $\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19. $(-t)^n f(t)$	$F^{(n)}(s)$