3.7 and 3.8: Mechanical and Electrical Vibrations Application Descriptions

In this sheet, we discuss the set up of two applications of second order constant homogeneous equations.

Application 1: Oscillating Spring (See the first figure in section 3.7)
A spring is attached to the ceiling and allowed to hang downward.
Let l be the natural length of a spring with no mass attached.
Let L be the distance beyond natural length it is stretched when an object of mass of m kg is attached.
In other words, $l + L$ is the distance from the ceiling when the object is at rest.
Let $u(t)$ be the displacement of the spring from rest (with positive downward) at time t.
We will move the object to a starting displacement $u(0) = u_0$ and push it with an initial velocity $u'(0) = v_0$ and study the resulting motion.
Forces:

- $F_g = w = mg$. (Force due to gravity)
  Another name for this is the 'weight'. It is always downward which we are calling positive.

- $F_s = -k(L + u(t))$. (Force due to the spring, *e.g.* restoring force)
  This is 'Hooke's Law' which says that the force is proportional to the distance from natural position. In this case $L + u$ is the distance from natural position. Note that if $L + u$ is positive, then this force will be negative (upward) and if $L + u$ is negative this force will be positive (downward).

- $F_d = -\gamma u'(t)$. (Force due to damping, *e.g.* friction force)
  This is one model for friction that assumes that the friction force it proportional to velocity and in the positive direction. Note that if $u'(t)$ is positive, then F_d is negative and if $u'(t)$ is negative, then F_d is positive. We used the same model earlier in the term for air resistance.

- $F_e = F(t) =$ 'some external force'.
  This can be any function (typically periodic) that describes an external force for any time t.

- Special Note: When the object is at rest (in other words when it is sitting with $u(0) = 0$ and $u'(0) = 0$) all the forces will add to zero. Which means that $mg - kL = 0$. Thus, in this situation we always have
$$w = mg = kL.$$

Newton's second law says that '(mass)(acceleration) = net force', so we have:

$$mu''(t) = mg - k(L + u(t)) - \gamma u'(t) + F(t) = mg - kL - ku(t) - \gamma u'(t) + F(t)$$

Thus,
$$mu'' + \gamma u' + ku = F(t).$$

Note:

- $m =$ 'the mass of the object':
  From above $w = mg$ and $m = \frac{w}{g}$.

- $\gamma =$ 'the damping constant' = 'the proportionality constant in the friction force'
  From above $F_d = -\gamma u'$ and $\gamma = -\frac{F_d}{u'}$.

- $k =$ 'the spring constant' = 'the proportionality constant in the spring force'
  From above $w = mg = kL$, so $k = \frac{w}{L} = \frac{mg}{L}$.

Comment about units:
In US standard units, the unit pounds (lbs) is a force unit. **Pounds (lbs) is NOT a mass unit**. Pounds is already weight, w, you don't need to multiply by gravity. However, in metric units the unit kilograms (kg) is a mass unit (it is NOT force unit), so you do have to multiply by, $g = 9.8$, in order to get the force unit of Newtons. Let me summarize the important unit facts below:

Type	Metric	US Standard
$m =$ Mass	kg	slugs (not commonly used)
$g =$ Accel. due to gravity on Earth	9.8 m/s^2	32 ft/s^2
$w = mg =$ Weight (Force)	N = Newtons	pounds = lbs
$u(t) =$ displacement	m = meters	ft = feet

Examples:

1. A mass weighing 3 kg stretches a spring 60 cm (0.06 meters) beyond natural length.
   The force due to resistance is 8 N when the upward velocity is 2 m/s (*i.e.* when $u' = -2$).
   The mass is given an initial displacement of 20 cm (0.02 meters) and is released (*i.e.* the initial velocity is zero). Assume there is no external forcing. Set up the differential equation and initial conditions for u.

   Solution:
   You are given $m = 3$ kg, $L = 0.06$ m, and we know $g = 9.8$ m/s^2.
   At rest we know $mg = kL$. Thus, $k = \frac{w}{L} = \frac{mg}{L} = \frac{3 \cdot 9.8}{0.06} = 490$ N/m.
   We also are told that $F_d = -\gamma u' = 8$ N when $u' = -2$ m/s. Thus, $\gamma = -\frac{F_d}{u'} = -\frac{8}{-2} = 4$ N·s/m.
   Therefore, $3u'' + 4u' + 490u = 0$, with $u(0) = 0.02$ and $u'(0) = 0$.

2. A mass weighing 8 lbs stretches a spring 2 in ($\frac{1}{6}$ ft) beyond natural length.
   The force due to resistance is 3 lbs when the upward velocity is 1 ft/s (*i.e.* when $u' = -1$).
   The mass is given an initial displacement of 6 in and an initial upward velocity of 2 ft/s.
   Assume there is no external forcing. Set up the differential equation and initial conditions for u.

   Solution:
   You are given $w = mg = 8$ lbs, $L = \frac{1}{6}$ ft, and we know $g = 32$ ft/s^2.
   Thus, $m = \frac{w}{g} = \frac{8}{32} = \frac{1}{4}$ lbs$\cdot$ s^2/ft (slugs).
   At rest we know $mg = kL$. Thus, $k = \frac{w}{L} = \frac{8}{1/6} = 48$ lbs/ft.
   We also are told that $F_d = -\gamma u' = 3$ lbs when $u' = -1$ ft/s. Thus, $\gamma = -\frac{F_d}{u'} = -\frac{3}{-1} = 3$ lbs·s/ft.
   Therefore, $\frac{1}{4}u'' + 3u' + 48u = 0$, with $u(0) = \frac{1}{2}$ and $u'(0) = -2$.

Application 2: Electrical Vibrations (see the last figure in section 3.7)
Consider the flow of electricity through a series circuit containing a resistor, an inductor, and a capacitor (called an RLC circuit). The total charge on the capacitor at time t is $Q = Q(t)$ in coulombs (C). We also define $I = I(t) = Q'(t)$ to be the current in the circuit at time t in amperes (A). Our goal will be to find the function $Q(t)$.
First, let me define some constants, variables and units.

Definitions and Kirchoff's circuit laws:

- Kirchoff's second law states: *In a closed circuit the impressed voltage is equal to the sum of the voltage drops in the rest of the circuit*
- We will let $E = E(t)$ be the impressed voltage in volts (V), which is the incoming voltage to the circuit.
- Laws of electricity:
    1. The voltage drop across the resistor is proportional to the current that flows through it.
    We write $RI = RQ'$, where R is the proportionality constant due to resistance.
    We call R the resistance with the unit ohms (Ω).
    2. The voltage drop across the capacitor is proportional to the total charge on the capacitor.
    Convention is to write $\frac{1}{C}Q$, where $\frac{1}{C}$ is proportionality constant due to the capacitor.
    We call C the capacitance with the unit farads (F).
    3. The voltage drop across the inductor is proportional to the derivative of the current.
    We write $LI' = LQ''$, where L is the proportionality constant due to the inductor.
    We call L the inductance with the unit henrys (H).
- The units are related as follows: $V = \Omega \cdot A = \frac{C}{F}$, and $\Omega = \frac{H}{s}$

Putting these laws together, we have

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

Examples:

1. A series circuit has a capacitor of 0.00003 F, a resistor of 200 Ω, and an inductor of 0.6 H.
There is no impressed voltage. The initial charge on the capacitor is 0.0001 C and there is no initial current. Set up the differential equation and initial conditions for the charge $Q(t)$.

    Solution: You are given $C = 0.00003$, $R = 200$, $L = 0.6$, and $E(t) = 0$.
    Therefore, $0.6Q'' + 200Q' + (1/0.00003)\,Q = 0$, with $Q(0) = 0.0001$ and $Q'(0) = 0$.

2. A series circuit has a capacitor of 0.0002 F and an inductor of 1.5 H (and no resistor).
There is no impressed voltage. The initial charge on the capacitor is 0.005 C and there is no initial current. Set up the differential equation and initial conditions for the charge $Q(t)$.

    Solution:You are given $C = 0.0002$, $R = 0$, $L = 1.5$, and $E(t) = 0$.
    Therefore, $1.5Q'' + (1/0.0002)\,Q = 0$, with $Q(0) = 0.005$ and $Q'(0) = 0$.