3.8: Analyzing Mechanical and Electrical Vibrations (Forced Vibrations)

In section 3.8 we are considering 'forced vibrations'. In other words, we are considering the nonhomogeneous equation with $F(t) \neq 0$ in this section. One of the most common/natural situations is a forcing function that is oscillating. We have shown how to write waves in the standard way $R\cos(\omega t - \delta)$. Thus, to keep the algebra and analysis simple, we will focus only on forcing functions of the form

$$F(t) = F_0 \cos(\omega t)$$

For the situation of forcing in the mass-spring system, the displacement from rest, $u(t)$, at time t satisfies:

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t),$$

where m is the mass, γ is the damping (friction) constant, and k is the spring constant (all these constants are positive).

Undamped Forced Vibrations: (The $\gamma = 0$ case)
If we assume there is no friction, then we are taking $\gamma = 0$. In which case we get:

$$mu'' + ku = F_0 \cos(\omega t).$$

As we noted in section 3.7, the homogeneous solution has the form $u_c(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$ where $\omega_0 = \sqrt{k/m}$. The particular solution will have the form $U(t) = A\cos(\omega t) + B\sin(\omega t)$ or the form $U(t) = At\cos(\omega t) + Bt\sin(\omega t)$ depending on whether $\omega \neq \omega_0$ or $\omega = \omega_0$. (Remember our undetermined coefficient discussion if you don't know why).

1. If $\omega \neq \omega_0$, then it turns out a particular solution has the form $U(t) = \frac{F_0}{m(w_0^2 - w^2)}\cos(\omega t)$.
   In this case, the general solution is

   $$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(w_0^2 - w^2)}\cos(\omega t)$$

   In electronics, this situation is used in what is called amplitude modulation. See a nice picture of this phenomenon in Figure 3.8.7 of the book.

2. If $\omega = \omega_0$, then it turns out a particular solution has the form $U(t) = \frac{F_0}{2m\omega_0} t\sin(\omega_0 t)$.
   In this case, the general solution is

   $$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t\sin(\omega_0 t)$$

   The function $t\sin(\omega_0 t)$ in the solution is **unbounded**! The amplitude of the wave keeps growing. There is never zero damping so this is a bit unrealistic, but this does illustrate that when $\omega \approx \omega_0$ and damping is very small, then the amplitude can get very large. This phenomenon is called **resonance**. We will discuss this again on the next page in the more general case.
   See a nice picture of this phenomenon in Figure 3.8.8 in the book.

Damped Forced Vibrations: (The $\gamma > 0$ case)
If $\gamma > 0$, then we have

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t).$$

The homogeneous solution has the form $u_c(t) = c_1 u_1(t) + c_2 u_2(t)$ where $u_1(t)$ and $u_2(t)$ are determined as we did in section 3.7. (Remember $\gamma > 2\sqrt{mk}$ gives overdamped, $\gamma = 2\sqrt{mk}$ gives critically damped, and $\gamma < 2\sqrt{mk}$ gives oscillations with decreasing amplitudes).
In all these cases when $\gamma \neq 0$, the particular solution will take the form $U(t) = A\cos(\omega t) + B\sin(\omega t)$. Thus, the general solution for undamped forced vibrations will always have the form

$$u(t) = (c_1 u_1(t) + c_2 u_2(t)) + (A\cos(\omega t) + B\sin(\omega t)) = u_c(t) + U(t)$$

Notes:

- The homogeneous solution, in this case, goes to zero as $t \to \infty$. That is, $\lim_{t\to\infty} u_c(t) = 0$.
- Since $u_c(t)$ dies out, we call it the **transient solution**. The transient solution allows us to meet the initial conditions, but in the long run the damping causes the transient solution to die out and the forcing takes over. The particular solution $U(t) = A\cos(\omega t) + B\sin(\omega t)$ is called the **steady state solution**, or **forced response**.
- Through substitution and lengthy algebra, you can find the coefficients in the particular solution. For analysis it is convenient to write the solution in the wave form $U(t) = R\cos(\omega t - \delta)$.
  We get $R = \frac{F_0}{\Delta}$, $\cos(\delta) = \frac{m(w_0^2 - w^2)}{\Delta}$, and $\sin(\delta) = \frac{\gamma\omega}{\Delta}$,
  where $\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$ and $\omega_0 = \sqrt{k/m}$.
  This is messy, but the first observation is that the steady state solution has frequency ω (which is the same as the forcing function). The second observation, with some work, is that the formula for amplitude, R, of the steady state solution can be rewritten as

  $$R = \frac{F_0}{k}\left(\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \frac{\gamma^2}{mk}\frac{\omega^2}{\omega_0^2}\right)^{-1/2}.$$

  Note: as $\omega \to 0$, $R \to \frac{F_0}{k}$, and as $\omega \to \infty$, $R \to 0$. In terms of ω, the maximum value of this function occurs when $\omega_{\text{max}} = \sqrt{\omega_0^2 - \gamma^2/(2m^2)}$ and at this value you get $R_{\text{max}} = \frac{F_0}{\gamma\omega_0\sqrt{1-\gamma^2/(4mk)}}$.

  Thus, if damping is very small (i.e. if γ is close to zero), then the maximum amplitude occurs when $\omega \approx \omega_0$. In which case the amplitude will be about $\frac{F_0}{\gamma\omega_0}$ which can be quite large (and it gets larger the closer γ gets to zero). This phenomenon is known as **resonance**.

  Some Side Comments: Resonance is something you have to worry about with designing buildings and bridges (you don't want the wind, or the wrong pattern of traffic, to cause resonance that makes your bridge oscillate so much that it collapses).
  When designing a RLC circuit resonance is what you want. For example, if the incoming (forcing) voltage comes from a weak signal you are getting on your car antennae, then you might want to be able to adjust the circuit frequency, ω_0, to match the incoming frequency (you design the circuit so that resistance, or inductance, or capatacitance can be adjusted with a dial). If you get these two frequencies close, then you can get resonance which will lead to a solution like the incoming signal but with a much higher amplitude. These concepts are essential in the sending and receiving of radio transmissions.