3.2: Linearity and the Wronskian

This section contains various theorems about existence and uniqueness for second order linear systems. In lecture, we emphasized linearity and the Wronskian (Theorems 3.2.2, 3.2.3, and 3.2.4). For now, I want you to only worry about these theorems (you should read the others for your own interest).

For these theorems, we are talking about **homogeneous** linear equations. Many of the theorems apply to any situation of the form $y'' + p(t)y' + q(t)y = 0$. Our immediate applications of these theorems (in 3.1, 3.3, and 3.4) will be concerned with the simpler case of constant coefficients ($ay'' + by' + cy = 0$), but the theorems hold in the general linear case as well.

Linearity/Superposition Theorem:
In general, if $y = y_1(t)$ and $y = y_2(t)$ are two solutions to $y'' + p(t)y' + q(t)y = 0$,
then $y = c_1y_1(t) + c_2y_2(t)$ is also a solution for any constants c_1 and c_2.

Notes about linearity/superposition:

1. In other words, the theorem says that a **linear combination** of any two solutions is also a solution. We say $c_1y_1(t) + c_2y_2(t)$ is a linear combinations of y_1 and y_2.

2. You can quickly prove this as follows:
   Since $y_1(t)$ is a solution, you must have $y_1'' + p(t)y_1' + q(t)y_1 = 0$.
   Since $y_2(t)$ is a solution, you must have $y_2'' + p(t)y_2' + q(t)y_2 = 0$.
   Now consider $y = c_1y_1(t) + c_2y_2(t)$. Taking derivatives we see that:

   $$\begin{aligned} y'' + p(t)y' + q(t)y &= (c_1y_1'' + c_2y_2'') + p(t)(c_1y_1' + c_2y_2') + q(t)(c_1y_1 + c_2y_2) \\ &= c_1(y_1'' + p(t)y_1' + q(t)y_1) + c_2(y_2'' + p(t)y_2' + q(t)y_2) \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0 \end{aligned}$$

   Thus, for any numbers c_1 and c_2 the function $y = c_1y_1(t) + c_2y_2(t)$ is also a solution!

3. For example: if $y_1(t) = e^{3t}$ and $y_2(t) = e^{-2t}$ are solutions to $y'' - y' - 6 = 0$,
   then $y(t) = c_1e^{3t} + c_2e^{-2t}$ is a solution for any numbers c_1 and c_2.

4. Another example: if $y_1(t) = e^{-4t}$ and $y_2(t) = te^{-4t}$ are solutions to $y'' - 8y' + 16 = 0$,
   then $y(t) = c_1e^{-4t} + c_2te^{-4t}$ is a solution for any numbers c_1 and c_2.

5. And another example: if $y_1(t) = \sin(7t)$ and $y_2(t) = \cos(7t)$ are solutions to $y'' + 49y = 0$,
   then $y(t) = c_1\sin(7t) + c_2\cos(7t)$ is a solution for any numbers c_1 and c_2.

6. Yet another example: if $y_1(t) = t$ and $y_2(t) = t\ln(t)$ are solutions to $t^2y'' - ty' + y = 0$,
   then $y(t) = c_1t + c_2t\ln(t)$ is a solutions for any numbers c_1 and c_2.

Wronskian:
Once you have two solutions and you have written $y(t) = c_1y_1(t) + c_2y_2(t)$, then we need to think about our initial conditions. First note that $y'(t) = c_1y_1'(t) + c_2y_2'(t)$.

Given **initial conditions**: $y(t_0) = y_0$ and $y'(t_0) = y_0'$. Substituting gives:

$$\begin{array}{rcl} y(t_0) = y_0 & \Rightarrow & c_1y_1(t_0) + c_2y_2(t_0) = y_0 \\ y'(t_0) = y_0 & \Rightarrow & c_1y_1'(t_0) + c_2y_2'(t_0) = y_0' \end{array}$$

This is a linear system of equation. See my review on two-by-two linear systems! From that discussion, you know that this has a **unique solution** for c_1 and c_2 if

$$\text{Wronskian determinant } = W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) \neq 0$$

In other words, if t_0 is a value where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_1' \end{vmatrix} \neq 0$, then there is a unique solution for c_1 and c_2.

Wronskian Fundmental Set of Solutions Theorem:
If $y = y_1(t)$ and $y = y_2(t)$ are solutions to $y'' + p(t)y' + q(t)y = 0$
AND if $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_1' \end{vmatrix} \neq 0$ for all valid values of t,
then we say y_1 and y_2 form a fundamental set of solutions.
In which case, **no matter the initial conditions** a unique solution for c_1 and c_2 will exist in the form $y = c_1y_1(t) + c_2y_2(t)$. In other words, if $W \neq 0$ for $y_1(t)$ and $y_2(t)$, then the solution $y = c_1y_1(t) + c_2y_2(t)$ is the **general solution** (meaning it contains all solutions).

Examples

1. For example: $y_1(t) = e^{3t}$ and $y_2(t) = e^{-2t}$ are solutions to $y'' - y' - 6 = 0$,
   and $W = \begin{vmatrix} e^{3t} & e^{-2t} \\ 3e^{3t} & -2e^{-2t} \end{vmatrix} = -5e^t \neq 0$.
   Thus, e^{3t} and e^{-2t} form a fundamental set of solutions.
   Thus, ALL solutions are in the form $y(t) = c_1e^{3t} + c_2e^{-2t}$ for some numbers c_1 and c_2.

2. And another example: $y_1(t) = \sin(t) - \cos(t)$ and $y_2(t) = \cos(t) - \sin(t)$ are solutions to $y'' + y = 0$,
   but $W = \begin{vmatrix} \sin(t) - \cos(t) & \cos(t) - \sin(t) \\ \cos(t) + \sin(t) & -\sin(t) - \cos(t) \end{vmatrix} = 0$ (it takes some expanding to check this).
   Thus, y_1 and y_2 do NOT form a fundamental set of solutions. The general answer CANNOT be written in the form $y = c_1(\sin(t) - \cos(t)) + c_2(\cos(t) - \sin(t))$. This is happening because $y_2(t) = -y_1(t)$, so the 'two' given solutions are actually multiples of each other!

3. The last example again: $y_1(t) = \sin(t)$ and $y_2(t) = \cos(t)$ are solutions to $y'' + y = 0$,
   and $W = \begin{vmatrix} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{vmatrix} = -\sin^2(t) - \cos^2(t) = -1 \neq 0$.
   Thus, y_1 and y_2 form a fundamental set of solutions.
   Thus, ALL solutions are in the form $y(t) = c_1\sin(t) + c_2\cos(t)$ for some numbers c_1 and c_2.