Skills Review: Partial Fractions

Motivation:

Before we talk about splitting up fractions, let's review how to combine fractions. You have been doing this since elementary school. To add fractions we need a common denominator.

For example, $\frac{2}{5} + \frac{7}{3} = \frac{6}{15} + \frac{35}{15} = \frac{41}{15}$. Notice how we multiplied the top and bottom in each fraction in order to get equivalent fractions with the same common denominator. The same procedure works for functions involving fractions.

For example, if you are given $f(x) = \frac{1}{x-2} + \frac{3}{x-5}$ and you want to combine into one fraction, then you write $f(x) = \frac{1}{x-2} + \frac{3}{x-5} = \frac{x-5}{(x-2)(x-5)} + \frac{3x-6}{(x-2)(x-5)} = \frac{4x-11}{(x-2)(x-5)}$.

As another example, $g(x) = \frac{x}{x^2+4} + \frac{2}{x+3} = \frac{x^2+3x}{(x^2+4)(x+3)} + \frac{2x^2+8}{(x^2+4)(x+3)} = \frac{3x^2+3x+8}{(x^2+4)(x+3)}$.

In calculus you learned that integration of rational functions is often easier if you decompose into partial fractions. In other words, you go in the opposite direction of the two examples shown above. That is, if you wanted to integrate $\frac{4x-11}{(x-2)(x-5)}$, then it is easier to work with the partial fraction form $\frac{1}{x-2} + \frac{3}{x-5}$. And if you want to integrate $g(x) = \frac{3x^2+3x+8}{(x^2+4)(x+3)}$, then it is easier to work with the partial fraction form $\frac{x}{x^2+4} + \frac{2}{x+3}$. The method of partial fraction decomposition is used to split up rational functions in this way. In this class, partial fractions will help us solve linear constant coefficient differential equation in a very systematic (algebraic) way.

The Method of Decomposing into Partial Fractions:

Given a rational function $\frac{N(x)}{D(x)} = \frac{\text{Numerator}}{\text{Denominator}}$

1. **Simplify/Divide, if needed**:
   If anything can be canceled, do it! If the degree (highest power) of the numerator is greater than or equal to the degree of the denominator, divide!
   For example: Given $\frac{x^2}{x+3}$, you can divide to rewrite it as $x - 3 + \frac{9}{x+3}$.

2. **Factor Denominator and Complete Squares, if needed**:
   Factor the denominator into linear terms and irreducible quadratics. For the irreducible quadratics, complete the square.
   For example: Given $\frac{1}{(x+1)(x^2+10x+28)}$, completing the square gives $\frac{1}{(x+1)((x+5)^2+3)}$.

3. **Write the form of the decomposition**:
   Distinct linear terms that look like $x - a$ decompose as $\frac{A}{x-a}$.
   Repeated linear terms that look like $(x-a)^3$ decompose as $\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3}$.
   Distinct irreducible quadratic terms that look like $x^2 + a^2$ decompose as $\frac{Ax+B}{x^2+a^2}$.
   Repeated irreducible quadratic terms that look like $(x^2+a^2)^2$ decompose as $\frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{(x^2+a^2)^2}$.
   Here is an example that has several different types:
   $$\frac{BLAH}{(x+1)(x-3)^2((x-2)^2+7)} = \frac{A}{x+1} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{D(x-2)+E}{(x-2)^2+7}$$

4. **Solve for all the constants**:
   Clear the denominators and equate the two sides. There are sometimes shortcuts (as mentioned in class and in the examples below). But it always works to equate coefficients and solve the corresponding equations until you get all the constants A, B, C, ...

5. **Write your decomposition**: You are done!

Examples:

- Give the partial fraction decomposition for $\frac{x^2+2}{x^3+3x^2+2x}$

  1. Simplify/Divide: Done! Degree of numerator smaller than denominator.
  2. Factor: $\frac{x^2+2}{x(x^2+3x+2)} = \frac{x^2+2}{x(x+1)(x+2)}$
  3. Form: $\frac{x^2+2}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$
  4. Solve for constants: Clearing denominators gives $x^2+2 = A(x+1)(x+2) + Bx(x+2) + Cx(x+1)$. Expanding gives: $x^2+2 = (A+B+C)x^2 + (3A+2B+C)x + 2A$. So $A+B+C = 1$, $3A+2B+C = 0$ and $2A = 2$. Combining these condition you can solve this to get A, B, and C.
     Shortcut: Note that instead of expanding you can plug in strategic values of x to solve for A, B, and C faster (or use the cover up method to be even faster). Here are some strategically chosen values of x:
     Plugging in $x = 0$ gives $2 = 2A$, so $A = 1$.
     Plugging in $x = -1$ gives $3 = -B$, so $B = -3$.
     Plugging in $x = -2$ gives $6 = 2C$, so $C = 3$.
  5. Done: $\frac{x^2+2}{x^3+3x^2+2x} = \frac{1}{x} + \frac{-3}{x+1} + \frac{3}{x+2}$.

- Give the partial fraction decomposition for $\frac{x+1}{x^3+4x^2+8x}$

  1. Simplify/Divide: Done! Degree of numerator smaller than denominator.
  2. Factor: $\frac{x+1}{x(x^2+4x+8)}$. The expression x^2+4x+8 is irreducible ($b^2 - 4ac = -16 < 0$). Completing the square gives $x^2+4x+4-4+8 = (x+2)^2+4$
  3. Form: $\frac{x+1}{x((x+2)^2+4)} = \frac{A}{x} + \frac{B(x+2)+C}{(x+2)^2+4}$
  4. Solve for constants: Expanding gives $x+1 = A((x+2)^2+4) + (B(x+2)+C)x$.
     Let's start by plugging strategic values:
     Plugging in $x = 0$ gives $1 = 8A$, so $A = 1/8$.
     Plugging in $x = -2$ gives $-1 = 4A - 2C$, so $-1 = \frac{1}{2} - 2C$, so $C = \frac{3}{4}$.
     Thus, we have $x+1 = \frac{1}{8}((x+2)^2+4) + \left(B(x+2) + \frac{3}{4}\right)x$. You can expand this all out if you wish. I personally suggest you just look for a coefficient that would involve B. In this case, how are you going to get an x^2 term? On the left-hand side the coefficient of x^2 is zero. On the right hand side you can see that the only x^2 terms you will get are $\frac{1}{8}x^2$ and Bx^2. Thus, $0 = \frac{1}{8} + B$, so $B = -\frac{1}{8}$.
  5. Done: $\frac{x+1}{x(x^2+4x+8)} = \frac{1/8}{x} + \frac{-(1/8)(x+2)+3/4}{(x+2)^2+4}$

- Give the partial fraction decomposition for $\frac{5}{(x+1)(x-2)^2}$

1. Simplify/Divide: Done! Degree of numerator smaller than denominator.
2. Factor: Done! $\frac{5}{(x+1)(x-2)^2}$.
3. Form: $\frac{5}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$
4. Solve for constants: Expanding gives
   $5 = A(x-2)^2 + B(x+1)(x-2)^2 + C(x+1)$.
   Let's start by plugging strategic values:
   Plugging in $x = -1$ gives $5 = 9A$, so $A = 5/9$.
   Plugging in $x = 2$ gives $5 = 3C$, so $C = 5/3$.
   Thus, we have $5 = \frac{5}{9}(x-2)^2 + B(x+1)(x-2)^2 + \frac{5}{3}(x+1)$. You can expand this all out if you wish. Again, I personally suggest you look for coefficients involving B. In this situation, looking at coefficients of x^2, we have $0 = \frac{5}{9}x^2 + Bx^2$, so $B = -\frac{5}{9}$.
5. Done: $\frac{5}{(x+1)(x-2)^2} = \frac{5/9}{x+1} + \frac{-5/9}{x-2} + \frac{5/3}{(x-2)^2}$.

- Give the partial fraction decomposition for $\frac{1}{(x^2+1)(x^2+4)}$

1. Simplify/Divide: Done! Degree of numerator smaller than denominator.
2. Factor: Done!
3. Form: $\frac{1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$
4. Solve for constants: Clearing the denominators gives
   $1 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$.
   No good strategic values to plug in (could plug in i or $2i$ if you want to try), so let's expand to get $1 = Ax^3 + Bx^2 + 4Ax + 4B + Cx^3 + Dx^2 + Cx + D$ Thus, $A + C = 0$, $B + D = 0$, $4A + C = 0$, and $4B + D = 1$.
   Combining $A + C = 0$ and $4A + C = 0$ gives $A = 0$ and $C = 0$.
   Combining $B + D = 0$ and $4B + D = 1$ gives $B = 1/3$ and $D = -1/3$.
5. Done: $\frac{1}{(x^2+1)(x^2+4)} = \frac{1/3}{x^2+1} + \frac{-1/3}{x^2+4}$

More Shortcuts: The methods from the previous pages are enough to complete all our problems. For your own interest, I mention some other methods below that can sometimes be faster. **This is for you own interest (not required)**.

1. **The Cover Up Method For Distinct Linear Terms** Consider our first example: $\frac{x^2+2}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$. If we think of multiplying by x and substituting $x = 0$, we get the same shortcut method from the previous page, but in a simper form:

   For A: 'Cover up' the x in the denominator on the left-hand side and plug in $x = 0$ to what remains to get $A = \frac{(0)^2+2}{(0+1)(0+2)} = 1$.
   For B: 'Cover up' the $x+1$ in the denominator on the left-hand side and plug in $x = -1$ to what remains to get $B = \frac{(-1)^2+2}{-1(-1+2)} = -3$.
   For C: 'Cover up' the $x+2$ in the denominator on the left-hand side and plug in $x = -2$ to what remains to get $C = \frac{(-2)^2+2}{-2(-2+1)} = 3$.
   Notice this matches up with the solution in our first example. This 'cover up' shortcut always works for distinct linear terms.

2. **More Cover Up Method** Consider our third example: $\frac{5}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$.
   For the distinct linear term:
   For A: 'Cover up' the $(x+1)$ in the denominator on the left-hand side and plug in $x = -1$ to what remains to get $A = \frac{5}{(-1-2)^2} = \frac{5}{9}$.
   If we think of multiplying by $(x-2)^2$ and then plug in $x = 2$, we get:
   For C: 'Cover up' the $(x-2)^2$ in the denominator on the left-hand side and plug in $x = 2$ to what remains to get $C = \frac{5}{2+1} = \frac{5}{3}$.
   There are methods to find B, but, in order to keep this discussion short, use the methods from the previous page to find B.

3. **Using Complex Numbers** If we allow complex numbers, then we can factor 'irreducible' quadratics as well. For example $x^2 + 1 = (x-i)(x+i)$. Then we can use the cover up method on those as well. Consider our last example: $\frac{1}{(x^2+1)(x^2+4)} = \frac{1}{(x-i)(x+i)(x-2i)(x+2i)}$.
   We can write $\frac{1}{(x^2+1)(x^2+4)} = \frac{1}{(x-i)(x+i)(x-2i)(x+2i)} = \frac{A}{x-i} + \frac{B}{x+i} + \frac{C}{x-2i} + \frac{D}{x+2i}$. The cover up method gives:
   For A: Cover up $x-i$ and plug in i, to get $A = \frac{1}{(2i)(-i)(3i)} = \frac{1}{6i} = \frac{-1}{6}i$.
   For B: Cover up $x+i$ and plug in $-i$, to get $A = \frac{1}{(-2i)(-3i)(i)} = \frac{1}{-6i} = \frac{1}{6}i$.
   For C: Cover up $x-2i$ and plug in $2i$, to get $A = \frac{1}{(i)(3i)(4i)} = \frac{1}{-12i} = \frac{1}{12}i$.
   For D: Cover up $x+2i$ and plug in $-2i$, to get $A = \frac{1}{(-3i)(-i)(-4i)} = \frac{1}{12i} = \frac{-1}{12}i$.
   Thus, we get $\frac{-1/6i}{x-i} + \frac{1/6i}{x+i} + \frac{1/12i}{x-2i} + \frac{-1/12i}{x+2i}$.
   Combining $\frac{-1/6i}{x-i} + \frac{1/6i}{x+i} = \frac{1/3}{x^2+1}$ and $\frac{-1/12i}{x-2i} + \frac{1/12i}{x+2i} = \frac{-1/3}{x^2+4}$ which matches our earlier solution.

There are even more shortcuts and observations. Some of these methods aren't necessarily shorter, I am just exposing you to more options. Again, you don't need all of this, the methods discuss in the four examples on the previous pages are enough to do all the problems in this course. Hopefully you found some of the additional facts on this page to be of interest to you.