Laplace Transform Fact Sheet

General and Important Facts:

	General Result	Examples
Definition :	$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\, dt$	
Linearity :	$\mathcal{L}\{c_1 y_1(t) + c_2 y_2(t)\} = c_1 \mathcal{L}\{y_1(t)\} + c_2 \mathcal{L}\{y_2(t)\}$	$\mathcal{L}\{3e^{2t} - 5t^2\} = 3\mathcal{L}\{e^{2t}\} - 5\mathcal{L}\{t^2\}$
Linearity :	$\mathcal{L}^{-1}\{c_1 F(s) + c_2 G(s)\} = c_1 \mathcal{L}^{-1}\{F(s)\} + c_2 \mathcal{L}^{-1}\{G(s)\}$	$\mathcal{L}^{-1}\{\frac{2}{s+1} - \frac{4}{s}\} = 2\mathcal{L}^{-1}\{\frac{1}{s+1}\} - 4\mathcal{L}^{-1}\{\frac{1}{s}\}$
1st Deriv. :	$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - f(0)$	
2nd Deriv. :	$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sf(0) - f'(0)$	
Exponentials :	$\mathcal{L}\{e^{ct} f(t)\}(s) = \mathcal{L}\{f(t)\}(s-c)$	$\mathcal{L}\{e^{5t}\sin(2t)\}(s) = \mathcal{L}\{\sin(2t)\}(s-5)$
Multiplied by t :	$\mathcal{L}\{tf(t)\}(s) = -\frac{d}{ds}\left(\mathcal{L}\{f(t)\}\right)$	$\mathcal{L}\{t\sin(3t)\} = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}$
Unit step :	$u_c(t) = \begin{cases} 0, & t < c; \\ 1, & t \geq c. \end{cases}$	$u_2(t) = \begin{cases} 0, & t < 2; \\ 1, & t \geq 2. \end{cases}$

Elementary Laplace Transform Table: Here n is a positive integer.

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	*Examples/Notes :*	
1	$\frac{1}{s}$	$\mathcal{L}\{6\} = 6\mathcal{L}\{1\} = \frac{6}{s}$	$\mathcal{L}^{-1}\{\frac{7}{s}\}(t) = 7$
e^{at}	$\frac{1}{s-a}$	$\mathcal{L}\{4e^{5t}\} = \frac{4}{s-5}$	$\mathcal{L}^{-1}\{\frac{9}{s+3}\}(t) = 9e^{-3t}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$	$\mathcal{L}\{7\cos\left(\frac{1}{2}t\right)\} = \frac{7s}{s^2+1/4}$	$\mathcal{L}^{-1}\{\frac{5s}{s^2+4}\}(t) = 5\cos(2t)$
$\sin(bt)$	$\frac{b}{s^2+b^2}$	$\mathcal{L}\{5\sin(3t)\} = \frac{15}{s^2+9}$	$\mathcal{L}^{-1}\{\frac{7}{s^2+4}\}(t) = \frac{7}{2}\sin(2t)$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$\mathcal{L}\{6e^{2t}\cos(t)\} = \frac{6(s-2)}{(s-2)^2+1}$	$\mathcal{L}^{-1}\{\frac{5(s+1)}{(s+1)^2+9}\}(t) = 5e^{-t}\cos(3t)$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$\mathcal{L}\{2e^{t}\sin(5t)\} = \frac{10}{(s-1)^2+25}$	$\mathcal{L}^{-1}\{\frac{11}{(s+2)^2+16}\}(t) = \frac{11}{4}e^{-2t}\sin(4t)$
t^n	$\frac{n!}{s^{n+1}}$	$\mathcal{L}\{t\} = \frac{1}{s^2}$, $\mathcal{L}\{t^3\} = \frac{3!}{s^4}$	$\mathcal{L}^{-1}\{\frac{7}{s^4}\}(t) = \frac{7}{3!}t^3 = \frac{7}{6}t^3$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$\mathcal{L}\{7t^2e^{8t}\} = \frac{7\cdot 2}{(s-8)^2} = \frac{14}{(s-8)^2}$	$\mathcal{L}^{-1}\{\frac{5}{(s+4)^3}\}(t) = \frac{5}{2}t^2e^{-4t}$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$\mathcal{L}\{9u_2(t)\} = \frac{9e^{-2s}}{s}$	$\mathcal{L}^{-1}\{5\frac{e^{-3s}}{s}\} = 5u_3(t)$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	$\mathcal{L}\{u_3(t)e^{2(t-3)}\} = e^{-3s}\mathcal{L}\{e^{2t}\} = \frac{e^{-3s}}{s-2}$	$\mathcal{L}^{-1}\{\frac{e^{-11s}}{s-6}\}(t) = u_{11}(t)e^{6(t-11)}$
$(e^{bt}+e^{-bt})/2 = \cosh(bt)$	$\frac{s}{s^2-b^2}$	$\mathcal{L}\{\cosh(2t)\} = \frac{s}{s^2-4}$	
$(e^{bt}-e^{-bt})/2 = \sinh(bt)$	$\frac{b}{s^2-b^2}$	$\mathcal{L}\{\sinh(3t)\} = \frac{3}{s^2-9}$	
$\delta_c(t)$	e^{-cs}	$\delta_c(t)$ is the unit impulse function at c	

Terminology/Convolution:
For all below consider the forced system: $my'' + \gamma y' + ky = f(t)$ with $y(0) = 0$ and $y'(0) = 0$.

Name	Definition	Comments
Transfer Function :	$G(s) = \frac{1}{ms^2+\gamma s+ks}$	
Impulse Response :	$g(t) = \mathcal{L}^{-1}\{G(s)\}$	also the solution to $my'' + \gamma y' + ky = \delta_0(t)$
Input :	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$ is the forcing function.
Output :	$Y(s) = G(s)F(s) = \mathcal{L}\{y(t)\}$	the LaPlace Transform of the solution.
Convolution :	$y(t) = \int_0^t g(t-s)f(s)\, ds$	a 'shortcut' to compute $\mathcal{L}^{-1}\{G(s)F(s)\}$

Laplace Transform Method:
To solve $ay'' + by' + cy = g(t)$, where $g(t)$ can be any forcing function (we even discuss how it can have discontinuities).

1. **Take the Laplace transform of both sides**.
   Since the transform is linear, we get $a\mathcal{L}\{y''\} + b\mathcal{L}\{y'\} + c\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$.

2. **Use the rules for the 1st and 2nd derivative and solve for $\mathcal{L}\{y\}$**.
   Since $\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - f(0)$ and $\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sf(0) - f'(0)$,
   we get $(as^2 + bs + c)\mathcal{L}\{y\} - (as + b)f(0) - af'(0) = \mathcal{L}\{g(t)\}$.
   Also replace $\mathcal{L}\{g(t)\}$ by its Laplace transform. Now solve for $\mathcal{L}\{y\}$.

3. **Partial Fractions**:
   Break up the expression you found into partial fractions.

4. **Look in the table for the inverse Laplace transform**:
   Look up the answers in the table.

Examples: Try these on your own before you look at the solutions (solutions on the next page).

1. Solve $y'' + 3y' - 4y = 0$ with $y(0) = 0$ and $y'(0) = 6$, using the Laplace transform.

2. Solve $y'' + 2y' + y = 0$ with $y(0) = 3$ and $y'(0) = 1$, using the Laplace transform.

3. Solve $y'' - y = e^{2t}$ with $y(0) = 0$ and $y'(0) = 1$, using the Laplace transform.

4. Solve $y'' + y = u_5(t)$ with $y(0) = 0$ and $y'(0) = 3$, using the Laplace transform.

Solutions to examples:

1. Solve $y'' + 3y' - 4y = 0$ with $y(0) = 0$ and $y'(0) = 6$, using the Laplace transform.
   (a) *Laplace Transform*: $\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} - 4\mathcal{L}\{y\} = \mathcal{L}\{0\}$.
   (b) *Use Rules and Solve*: $s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 3s\mathcal{L}\{y\} - 3y(0) - 4\mathcal{L}\{y\} = 0$,
   which becomes: $(s^2 + 3s - 4)\mathcal{L}\{y\} - 6 = 0$.
   Solving for $\mathcal{L}\{y\}$ gives: $\mathcal{L}\{y\} = \frac{6}{s^2+3s-4}$.
   (c) *Partial Fractions*: $\frac{6}{s^2+3s-4} = \frac{6}{(s+4)(s-1)} = \frac{A}{s+4} + \frac{B}{s-1}$ and you find $A = -\frac{6}{5}$, $B = \frac{6}{5}$.
   (d) *Inverse Laplace transform*:
   The solution is: $y(t) = \mathcal{L}^{-1}\{\frac{-6/5}{s+4} + \frac{6/5}{s-1}\} = -\frac{6}{5}e^{-4t} + \frac{6}{5}e^{t}$.

2. Solve $y'' + 2y' + y = 0$ with $y(0) = 3$ and $y'(0) = 1$, using the Laplace transform.
   (a) *Laplace Transform*: $\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{0\}$.
   (b) *Use Rules and Solve*: $s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2s\mathcal{L}\{y\} - 2y(0) + \mathcal{L}\{y\} = 0$,
   which becomes: $(s^2 + 2s + 1)\mathcal{L}\{y\} - (7 + 3s) = 0$.
   Solving for $\mathcal{L}\{y\}$ gives: $\mathcal{L}\{y\} = \frac{3s+7}{s^2+2s+1}$.
   (c) *Partial Fractions*: $\frac{3s+7}{s^2+2s+1} = \frac{3s+7}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$ and you find $A = 3$, $B = 4$.
   (d) *Inverse Laplace transform*:
   The solution is: $y(t) = \mathcal{L}^{-1}\{\frac{3}{s+1} + \frac{4}{(s+1)^2}\} = 3e^{-t} + 4te^{-t}$.

3. Solve $y'' - y = e^{2t}$ with $y(0) = 0$ and $y'(0) = 1$, using the Laplace transform.
   (a) *Laplace Transform*: $\mathcal{L}\{y''\} - \mathcal{L}\{y\} = \mathcal{L}\{e^{2t}\}$.
   (b) *Use Rules and Solve*: $s^2\mathcal{L}\{y\} - sy(0) - y'(0) - \mathcal{L}\{y\} = \frac{1}{s-2}$,
   which becomes: $(s^2 - 1)\mathcal{L}\{y\} - 1 = \frac{1}{s-2}$.
   Solving for $\mathcal{L}\{y\}$ gives: $\mathcal{L}\{y\} = \frac{1}{(s-2)(s^2-1)} + \frac{1}{s^2-1}$.
   (c) *Partial Fractions*: $\frac{1}{(s-2)(s^2-1)} = \frac{1}{(s-2)(s-1)(s+1)} = \frac{A}{s-2} + \frac{B}{s-1} + \frac{C}{s+1}$ and you find $A = \frac{1}{3}$, $B = -\frac{1}{2}$, $C = \frac{1}{6}$.
   And $\frac{1}{s^2-1} = \frac{1}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1}$ and you find $A = \frac{1}{2}$ and $B = -\frac{1}{2}$.
   (d) *Inverse Laplace transform*:
   The solution is: $y(t) = \mathcal{L}^{-1}\{\frac{1/3}{s-2} - \frac{1/2}{s-1} + \frac{1/6}{s+1} + \frac{1/2}{s-1} - \frac{1/2}{s+1}\} = \frac{1}{3}e^{2t} + \frac{1}{6}e^{-t} - \frac{1}{2}e^{-t}$.
   Thus, $y(t) = \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t}$.

4. Solve $y'' + y = u_5(t)$ with $y(0) = 0$ and $y'(0) = 3$, using the Laplace transform.
   (a) *Laplace Transform*: $\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{u_5(t)\}$.
   (b) *Use Rules and Solve*: $s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \mathcal{L}\{y\} = \frac{e^{-5s}}{s}$,
   which becomes: $(s^2 + 1)\mathcal{L}\{y\} - 3 = \frac{e^{-5s}}{s}$.
   Solving for $\mathcal{L}\{y\}$ gives: $\mathcal{L}\{y\} = \frac{e^{-5s}}{s(s^2+1)} + \frac{3}{s^2+1}$.
   (c) *Partial Fractions*: $\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$ and you find $A = 1$, $B = -1$, and $C = 0$.
   (d) *Inverse Laplace transform*:
   The solution is: $y(t) = \mathcal{L}^{-1}\{e^{-5s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + 3\frac{1}{s^2+1}\}$, which is the same as
   $u_5(t)\mathcal{L}^{-1}\{\frac{1}{s} - \frac{s}{s^2+1}\}(t-5) + 3\mathcal{L}^{-1}\{\frac{1}{s^2+1}\}(t)$, and we get
   $y(t) = u_5(t)(1 - \cos(t-5)) + 3\sin(t)$.