Laplace Transform Example

Here is a example very similar a homework problem.

Solve $y'' + 2y' + 5y = 29e^{4t}$ with $y(0) = 3$ and $y'(0) = 0$, using the Laplace transform.

1. *Laplace Transform*: $\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = 29\mathcal{L}\{e^{4t}\}$.

2. *Use Rules and Solve for* $\mathcal{L}\{y\}$: $(s^2\mathcal{L}\{y\} - sy(0) - y'(0)) + 2(s\mathcal{L}\{y\} - y(0)) + 5\mathcal{L}\{y\} = \dfrac{29}{s-4}$.

   Thus, $(s^2 + 2s + 5)\mathcal{L}\{y\} - s(3) - (0) - 2(3) = \dfrac{29}{s-4}$,
   which gives
   $$\mathcal{L}\{y\} = \frac{29}{(s-4)(s^2+2s+5)} + \frac{3s+6}{s^2+2s+5}.$$

3. *Partial Fractions*:

   - Since $s^2 + 2s + 5$ is irreducible, complete the square: $s^2 + 2s + 5 = s^2 + 2s + 1 - 1 + 5 = (s+1)^2 + 4$.
   - Partial Fraction Decomposition of the First Term:
     $$\frac{29}{(s-4)((s+1)^2+4)} = \frac{A}{s-4} + \frac{B(s+1)+C}{(s+1)^2+4}$$
     Expanding gives $29 = A((s+1)^2 + 4) + (B(s+1) + C)(s-4)$.
     Plugging in $s = 4$ gives $A = 1$. Comparing Coefficients gives:
     The s^2 coefficient: $A + B = 0$, so $B = -1$.
     The constant term: $5A - 4B - 4C = 29$, so $4C = 5A - 4B - 29 = -20$, giving $C = -5$.
   - Partial Fraction Decomposition of the Second Term:
     $$\frac{3s+6}{(s+1)^2+4} = \frac{A(s+1)+B}{(s+1)^2+4}$$
     Expanding gives $3s + 6 = A(s+1) + B$, so $A = 3$ and $A + B = 6$ giving $B = 3$.

   Thus,
   $$\mathcal{L}\{y\} = \frac{1}{s-4} + \frac{-(s+1)-5}{(s+1)^2+4} + \frac{3(s+1)+3}{(s+1)^2+4} = \frac{1}{s-4} + \frac{2(s+1)-2}{(s+1)^2+4}.$$

4. *Inverse Laplace transform*:
   $$\mathcal{L}\{y\} = \frac{1}{s-4} + 2\,\frac{(s+1)}{(s+1)^2+4} - 2\,\frac{1}{(s+1)^2+4}.$$
   $$y = \mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} + 2\mathcal{L}^{-1}\left\{\frac{(s+1)}{(s+1)^2+4}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}.$$
   Using the inverse Laplace tranform table:
   $$y(t) = e^{4t} + 2e^{-t}\cos(2t) - e^{-t}\sin(2t).$$