Exam 2 Review

This review sheet contains this cover page (a checklist of topics from Chapters 3).

Chapter 3: Second Order Equations

- 3.1, 3.3, 3.4: Homogeneous Equations. Solve $ar^2 + br + c = 0$.
  Then $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, $y = c_1 e^{rt} + c_2 t e^{rt}$, or $y = c_1 e^{\lambda t}\cos(\omega t) + c_2 e^{\lambda t}\sin(\omega t)$ depending on roots.
- 3.4: Reduction of Order: Given one solution $y_1(t)$, write $y(t) = u(t)y_1(t)$ and substitute into the differential equation. Then solve for $u(t)$. The general solution is $y(t) = u(t)y_1(t)$.
- 3.5: Nonhomogeneous Equations. Key observation: If $y(t)$ and $Y(t)$ are any two solutions, then $y(t) - Y(t)$ is a solutions to the corresponding homogeneous equation. Thus, every solution will have the form: $y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$.
  Step 1: Find a fundamental set of solutions to the corresponding homogeneous equation.
  Step 2: Find a particular solution to the given equation using **undetermined coefficients**.
- 3.7, 3.8 Set Up:
  For mass-spring systems: A spring hangs down from the ceiling. A mass is attached to the spring and it comes to rest at a distance of L from natural length (this is called the resting position or equilibrium position and it is when $u = 0$). The mass is pulled to an initial displacement of $u(0)$ and set into motion with an initial velocity of $u'(0)$. Let $u(t)$ be the displacement from rest.
  By discussing the forces, we derived the second order system:

  $$mu'' + \gamma u' + ku = F(t) \text{ , where}$$

  - $F(t)$ = external forcing function
  - m = 'the mass of the object', we know $w = mg$ and $m = \frac{w}{g}$
  - γ = 'the damping constant', we know $F_d = -\gamma u'$ and $\gamma = -\frac{F_d}{u'}$
  - k = 'the spring constant', we know $w = mg = kL$, so $k = \frac{w}{L} = \frac{mg}{L}$

  If you are worried about units, all you needed in the homework was:
  $g = 32$ ft/s^2 $= 9.8$ m/s^2, 100 cm = 1m, 12 in = 1 ft,
  and these are the only conversions you'll need to know for my exam.

  (NOT ON TEST) 3.2: Linearity/Fundamental Sets. If $y_1(t)$ and $y_2(t)$ are solutions to $y'' + p(t)y' + q(t)y = 0$ and the Wronskian $W(y_1, y_2)$ is not zero at the initial conditions, then there is a unique solution of the form $y = c_1 y_1(t) + c_2 y_2(t)$. This is NOT on the the test, I only mention it as this was essential to our discussion all chapter.

  (NOT ON TEST) *For an RLC circuit*: Let $Q(t)$ be the total charge on the capacitor in coulumbs (C). We have: $LQ'' + RQ' + \frac{1}{C}Q = E(t)$, where $E(t)$ is the impressed voltages in volts (V);
  R is the resistance in ohms (Ω);
  C is the capacitance in farads (F);
  L is the inductance in henrys (H). This may end up being an important application for some of you in engineering and you will see this on some old test, but we are NOT covering this topic this quarter and it will NOT be on our test.

- 3.7 Analysis: 'Free Vibrations' ($F(t) = 0$)

  1. The $F(t) = 0$ and $\gamma = 0$ case: $u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) = R\cos(\omega_0 t - \delta)$. Thus, the solution is a cosine wave with the following properties:
     The **natural frequency** is $\omega_0 = \sqrt{k/m}$ radians/second;
     The **period** is $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{m/k}$ seconds/wave;
     The **amplitude** is $R = \sqrt{c_1^2 + c_2^2}$. The **phase angle** is δ which is the equivalent angle at which the cosine wave starts ($R\cos(\delta) = c_1$ and $R\sin(\delta) = c_2$).
  2. The $F(t) = 0$ and $\gamma > 0$ case:
     $\gamma > 2\sqrt{km} \Rightarrow y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, with both roots negative (overdamped).
     $\gamma = 2\sqrt{km} \Rightarrow y = c_1 e^{rt} + c_2 t e^{rt}$, with one negative root (critically damped).
     $\gamma < 2\sqrt{km} \Rightarrow y = e^{\lambda t}(c_1 \cos(\mu t) + c_2 \sin(\mu t)) = Re^{\lambda t}\cos(\mu t - \delta)$.
     In this last case, we say the **quasi-frequency** is $\mu = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}$ radians/second;
     The **quasi-period** is $T = \frac{2\pi}{\mu}$ seconds/wave;
     The 'amplitude' is not constant, it is given by $Re^{\lambda t}$ which will always go to zero as $t \to \infty$ (for all damped cases).

- 3.8 Analysis: 'Force Vibrations.' Consider the forcing function $F(t) = F_0 \cos(\omega t)$.

  1. The $\gamma = 0$ case:
     Homogeneous solution: $u_c(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$ where $\omega_0 = \sqrt{k/m}$.
     Particular solution:
     - $\omega \neq \omega_0 \Rightarrow U(t) = A\cos(\omega t) + B\sin(\omega t) = \frac{F_0}{m(w_0^2 - w^2)}\cos(\omega t)$ (*Beats*)
     - $\omega = \omega_0 \Rightarrow U(t) = At\cos(\omega t) + Bt\sin(\omega t) = \frac{F_0}{2m\omega_0} t\sin(\omega_0 t)$ (*Resonance*)
  2. The $\gamma > 0$ case:
     Homogeneous solutions: See discussion in 3.7.
     Particular solution: $U(t) = A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta)$.
     Thus, the general solution for undamped forced vibrations will always have the form
     $u(t) = (c_1 u_1(t) + c_2 u_2(t)) + (A\cos(\omega t) + B\sin(\omega t)) = u_c(t) + U(t)$.
  3. The function $u_c(t)$ is called the **transient solution** (it dies out).
     The particular solution $U(t) = A\cos(\omega t) + B\sin(\omega t)$ is called the **steady state solution**, or **forced response**.
  4. If damping is very small (i.e. if γ is close to zero), then the maximum amplitude occurs when $\omega \approx \omega_0$. In which case the amplitude will be about $\frac{F_0}{\gamma\omega_0}$ which can be quite large (and it gets larger the closer γ gets to zero). This phenomenon is known as **resonance**.

- Other skills:

  - Solving two-by-two systems (when solving for initial conditions).
  - Working with complex numbers (when we used Euler's formula).
  - Working with cosine and sine (when we wrote it as one wave or two waves as a product).