Exam 1 Short Problem Practice

The following pages contains a random list of short problems from old exams pertaining to material from chapters 1 and 2. This is just to give you a taste of some of the types of questions I may ask on exam 1. This should not be your only form of studying. You must study:

- All the homework - be ready if I ask any questions like any of the homework questions (the only difference will be that exam question solutions will fit in half a page, so the algebra/integrations won't be as long as a couple of the really messy homework problems).
- Look at several old exams - start with the ones that have my name on it, then try some from the department archive, then try others from my archive.

1. (2.1) Find the general explicit solution to $\frac{dy}{dx} = \frac{x^3 - 2y}{x}$ for $x > 0$.

2. (2.2) Find the explicit solution to $\frac{dy}{dx} = e^{x+y}$ with $y(0) = 0$.

3. (2.3) Let $P(t)$ be the size of a population of fish, with t measured in days. The rate of population increase is proportional to the size of the population. The population increases by 2% per day. Furthermore, 1000 fish are harvested each day. Write down a differential equation for the number of fish in the population. (just set up, don't solve)

4. (2.3) A local pond has an initial volume of 5,000 liters. Water enters the pond through a stream at 400 liters/day and water leaves the pond through a different stream at 600 liters/day. Salt, from the salting of roads, gets into the incoming stream with a concentration of 0.03 kg/liter. Initially, the pond has no salt. Let $y(t)$ be the amount of salt in the pond at time t days. Give the differential equation and initial condition (just set up, don't solve).

5. (2.5) Find all equilibrium solutions for $\frac{dy}{dt} = (y^2 - 9)y^2$ and classify each as stable, unstable or semistable.

6. (2.7) Given $\frac{dy}{dt} = 1.5 - 0.5\sqrt{y}$, and $y(0) = 4$. Use Euler's method with step size 0.5 to approximate the value of $y(1)$.

Solutions on the next page

Solutions:

1. Linear!

   - Rewrite as $\dfrac{dy}{dx} = \dfrac{x^3 - 2y}{x} = x^2 - \dfrac{2}{x}y$, so $\dfrac{dy}{dx} + \dfrac{2}{x}y = x^2$.
   - The integrating factor is $\mu(x) = e^{\int 2/x\,dx} = e^{2\ln(x)} = x^2$. Note that $2\ln(x) = \ln(x^2)$.
   - Multiplying by the integrating factor and rewriting gives $\dfrac{d}{dx}(x^2 y) = x^4$.
   - Integrating $x^2 y = \dfrac{1}{5}x^5 + C$.

   Thus, $y(x) = \dfrac{1}{5}x^3 + \dfrac{C}{x^2}$.

2. Not linear, try separable!

   - Rewrite as $\dfrac{dy}{dx} = e^x e^y$ so $e^{-y}dy = e^x dx$.
   - Integrating gives $-e^{-y} = e^x + C$.
   - Solving for y gives $e^{-y} = -e^x - C$, so $y(x) = -\ln(-e^x - C)$.
   - The initial condition gives $0 = -\ln(-1 - C)$, which gives $1 = -1 - C$, so $C = -2$.

   Thus, $y(x) = -\ln(-e^x + 2)$.

3. Factoring $(y+3)(y-3)y^2 = 0$ and solving give eq. solns $y(t) = -3$, $y(t) = 0$ and $y(t) = 3$.

   - For $y < -3$: $(y^2 - 9)y^2$ is positive.
   - For $-3 < y < 0$: $(y^2 - 9)y^2$ is negative.
   - For $0 < y < 3$: $(y^2 - 9)y^2$ is negative.
   - For $3 < y$: $(y^2 - 9)y^2$ is positive.

   Thus, $y(t) = -3$ is STABLE, $y(t) = 0$ is SEMISTABLE, and $y(t) = 3$ is UNSTABLE.

4. We can directly translate the statement of the problem as $\dfrac{dP}{dt} = 0.02P - 1000$.

5. First find the volume of the pond after t days, $V(t) = 5000 + (400 - 600)t = 5000 - 200t$ liters. Next find the kg/day entering and kg/day exiting the pond.

$$\frac{dy}{dt} = (0.03)(400) - \left(\frac{y}{5000 - 200t}\right) \cdot (600) \text{ with} y(0) = 0.$$

6. Let $f(y,t) = 1.5 - 0.5\sqrt{y}$ and $h = 0.5$. The key formula in Euler's method is $y_{k+1} = y_k + f(y_k, t_k)h$.

   - $t_0 = 0$, $y_0 = 4$
     - slope $= f(4, 0) = 1.5 - 0.5\sqrt{4} = 0.5$
     - Thus, $y_1 = y_0 + f(4, 0)h = 4 + 0.5 \cdot 0.5 = 4.25$
   - $t_1 = 0.5$, $y_1 = 4.25$
     - slope $= f(4.25, 0.5) = 1.5 - 0.5\sqrt{4.25} \approx 0.46922$
     - Thus, $y_1 = y_0 + f(4, 0)h = 4.25 + 0.46822 \cdot 0.5 \approx 4.4846$