1. (12 points)

(a) Find the **equation of the plane** containing the two lines given by the parametric equations

$$L1: \begin{cases} x = 7 - 2t \\ y = 5 + t \\ z = 8 \end{cases} \qquad L2: \begin{cases} x = 7 + 4t \\ y = 5 - 3t \\ z = 8 + t \end{cases}$$

DIRECTION VECTOR FOR L1: $\vec{v}_1 = \langle -2, 1, 0 \rangle$

DIRECTION VECTOR FOR L2: $\vec{v}_2 = \langle 4, -3, 1 \rangle$

$\vec{v}_1$ and $\vec{v}_2$ are both parallel to the plane we want to find. So we obtain a normal by

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & 0 \\ 4 & -3 & 1 \end{vmatrix}$$

$$= \langle 1-0, 0--2, 6-4 \rangle = \langle 1, 2, 2 \rangle$$

← check: $\vec{n} \cdot \vec{v}_1 = 0$ ✓
$\vec{n} \cdot \vec{v}_2 = 0$ ✓

ASIDE: We see that the lines do intersect at (7, 5, 8). If the lines didn't intersect the question would not make sense as written.

$\vec{r}_0 = \langle 7, 5, 8 \rangle$

PLANE:
$$\boxed{\begin{gathered} \langle 1,2,2 \rangle \cdot (\langle x,y,z \rangle - \langle 7,5,8 \rangle) = 0 \\ (x-7) + 2(y-5) + 2(z-8) = 0 \\ x + 2y + 2z - 33 = 0 \end{gathered}}$$

(b) Consider the line, L, that is orthogonal to the plane $x - z + 7 = 0$ and through the point $(0, 1, 4)$. Find **an equation for the line**, then find all points where the line intersects the surface $z = x^2 + 2y^2$.

A normal for the plane is $\langle 1, 0, -1 \rangle$. Since the line, L, is orthogonal to the plane, $\vec{v} = \langle 1, 0, -1 \rangle$ gives a direction vector for the line.

$\vec{r}_0 = \langle 0, 1, 4 \rangle \qquad \vec{v} = \langle 1, 0, -1 \rangle$

LINE: $\langle x, y, z \rangle = \langle 0, 1, 4 \rangle + t\langle 1, 0, -1 \rangle$
$$\left\{ \boxed{\begin{aligned} x &= 0 + t \\ y &= 1 \\ z &= 4 - t \end{aligned}} \right.$$

INTERSECTION: $z = x^2 + 2y^2 \Rightarrow 4 - t = t^2 + 2(1)^2$

$0 = t^2 + t - 2$

$0 = (t+2)(t-1)$

$t = -2, \ t = 1$

$$\boxed{\begin{aligned} t=-2 &: (x,y,z) = (-3,1,6) \\ t=1 &: (x,y,z) = (1,1,3) \end{aligned}}$$

2. (5 pts) Consider the surface $z = x^2 + 2y^2$.

(a) Describe the traces parallel to the given plane (no work needed, just circle your answers).

i. Parallel to the yz-plane (when x is fixed):
PARABOLAS CIRCLES ELLIPSES HYPERBOLAS NONE OF THESE

ii. Parallel to the xz-plane (when y is fixed):
PARABOLAS CIRCLES ELLIPSES HYPERBOLAS NONE OF THESE

iii. Parallel to the xy-plane (when z is fixed, $z > 0$):
PARABOLAS CIRCLES ELLIPSES HYPERBOLAS NONE OF THESE

(b) Clearly circle the name of the surface given by $z = x^2 + 2y^2$:

CONE	SPHERE	ELLIPSOID
PARABOLIC CYLINDER	CIRCULAR CYLINDER	ELLIPTICAL CYLINDER
HYPERBOLIC CYLINDER	HYPERBOLOID	CIRCULAR PARABOLOID
ELLIPTIC PARABOLOID	HYPERBOLIC PARABOLOID	NONE OF THESE

3. (10 points) Olivo is running on a path. His location (x, y) (each in feet) at time t seconds is given by the vector function

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle \cos(\pi t),\ \sin(3\pi t) \rangle .$$

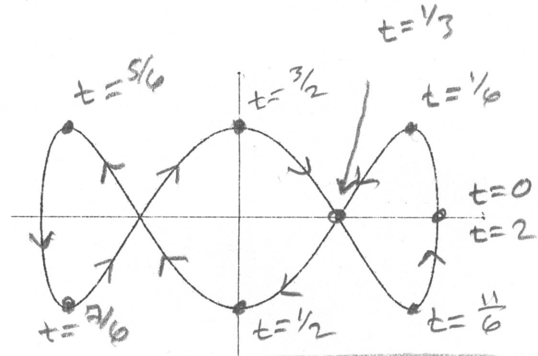
(a) Find the equation for the tangent line at $t = 1/3$.

$x(\tfrac{1}{3}) = \cos(\tfrac{\pi}{3}) = \tfrac{1}{2}$

$y(\tfrac{1}{3}) = \sin(\pi) = 0$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3\pi \cos(3\pi t)}{-\pi \sin(\pi t)}$

$\left.\dfrac{dy}{dx}\right|_{t=1/3} = \dfrac{3\,(-1)}{-\sqrt{3}/2} = \dfrac{6}{\sqrt{3}} = 2\sqrt{3}$



$$\boxed{y = \frac{6}{\sqrt{3}}\left(x - \tfrac{1}{2}\right) + 0 = 2\sqrt{3}\left(x - \tfrac{1}{2}\right) = 2\sqrt{3}\,x - \sqrt{3}}$$

← all correct →

(b) Find all three values of x at which the path has horizontal tangents.

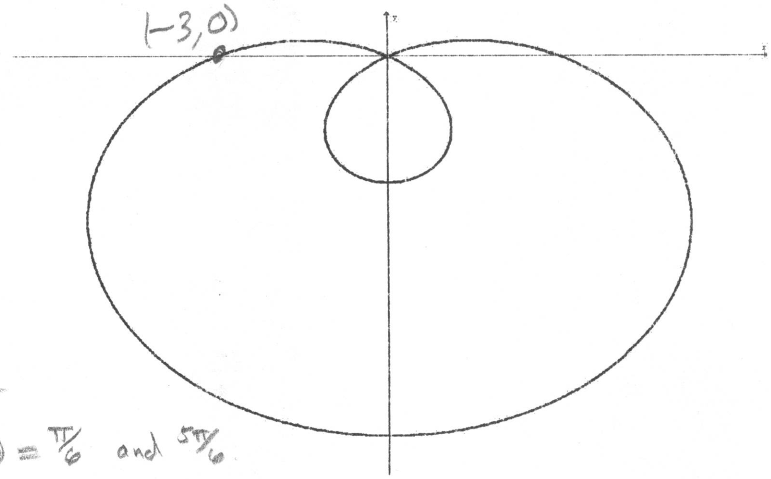
$\dfrac{dy}{dt} = 0\ ? \Rightarrow 3\pi \cos(3\pi t) = 0$

$3\pi t = \pi/2$ or $\frac{3\pi}{2}$ or $\frac{5\pi}{2}$ or $\frac{(2k+1)\pi}{2}$ in general for any integer k

$\Rightarrow t = \frac{1}{6}$ or $\frac{1}{2}$ or $\frac{5}{6}$ or $\frac{(2k+1)}{6}$

$$\boxed{\begin{aligned} x(\tfrac{1}{6}) &= \cos(\tfrac{\pi}{6}) = \tfrac{\sqrt{3}}{2} \\ x(\tfrac{1}{2}) &= \cos(\tfrac{\pi}{2}) = 0 \\ x(\tfrac{5}{6}) &= \cos(\tfrac{5\pi}{6}) = -\tfrac{\sqrt{3}}{2} \end{aligned}}$$

4. (11 pts) Consider the polar curve given by the equation $r = 3 - 6\sin(\theta)$. The graph of the curve is given below.



(a) Give the value of all y-intercepts.

Want to know when

$x = r\cos(\theta) \overset{?}{=} 0$

which happens ① when $r=0$

② when $\theta = \pi/2$

③ when $\theta = 3\pi/2$

① $r=0$ does occur when $\theta = \pi/6$ and $5\pi/6$.

So $\boxed{y=0}$ is one y-intercept.

② $\theta = \pi/2 \Rightarrow r = 3 - 6\sin(\pi/2) = -3 \Rightarrow y = r\sin(\theta) = -3$. $\boxed{y=-3}$

③ $\theta = \frac{3\pi}{2} \Rightarrow r = 3 - 6\sin(3\pi/2) = 9 \Rightarrow y = r\sin(\theta) = -9$ $\boxed{y=-9}$

(b) Find the equation for the tangent line at the point on the curve corresponding to $\theta = \pi$. (Give your answer in the form $y = mx + b$.)

$\theta = \pi \Rightarrow r = 3 - 6\sin(\pi) = 3$ $\begin{cases} x = r\cos(\theta) = -3 \\ y = r\sin(\theta) = 0 \end{cases}$

$x = r\cos(\theta) = (3 - 6\sin(\theta))\cos(\theta)$

$y = r\sin(\theta) = (3 - 6\sin(\theta))\sin(\theta)$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-6\cos(\theta)\sin(\theta) + (3 - 6\sin(\theta))\cos(\theta)}{-6\cos(\theta)\cos(\theta) - (3 - 6\sin(\theta))\sin(\theta)}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{-6(-1)(0) + (3)(-1)}{-6(-1)(-1) - (3)(0)} = \frac{-3}{-6} = \frac{1}{2} = \text{slope}$$

$$\boxed{y = \frac{1}{2}(x - -3) + 0 = \frac{1}{2}(x+3) = \frac{1}{2}x + \frac{3}{2}}$$

5. (12 points) The motion of a particular fly in three-dimensions is described by the vector position function $\mathbf{r}(t) = \langle t^2, t-4, -8+32\sqrt{4+t}\rangle$.

$16(4+t)^{-1/2}$

(a) Find the curvature at $t = 0$.

$\vec{r}\,'(t) = \langle 2t, 1, \frac{16}{\sqrt{4+t}} \rangle$ $\quad$ $\vec{r}\,''(t) = \langle 2, 0, \frac{-8}{(4+t)^{3/2}} \rangle$

$\vec{r}\,'(0) = \langle 0, 1, 8 \rangle$ $\quad$ $\vec{r}\,''(0) = \langle 2, 0, -1 \rangle$

$$\vec{r}\,'(0) \times \vec{r}\,''(0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 8 \\ 2 & 0 & -1 \end{vmatrix} = \langle -1-0, 16-0, 0-2 \rangle$$
$$= \langle -1, 16, -2 \rangle$$

$$\kappa(0) = \frac{|\vec{r}\,'(0) \times \vec{r}\,''(0)|}{|\vec{r}\,'(0)|^3} = \frac{\sqrt{1^2+16^2+2^2}}{(\sqrt{0^2+1^2+8^2})^3} = \boxed{\frac{\sqrt{261}}{65^{3/2}}}$$

(b) Find the location, (x, y, z), where the tangent line to the curve at $t = -3$ intersects the xy-plane.

$\vec{r}(-3) = \langle 9, -7, 24 \rangle$

$\vec{r}\,'(-3) = \langle -6, 1, 16 \rangle$

TANGENT LINE:

$\langle x, y, z \rangle = \langle 9, -7, 24 \rangle + t\langle -6, 1, 16 \rangle$

$x = 9 - 6t$

$y = -7 + t$

$z = 24 + 16t$

xy-plane $\Rightarrow z = 0 \Rightarrow$ $\quad$ $0 = 24 + 16t \Rightarrow t = \frac{-24}{16} = -\frac{3}{2}$

$x = 9 - 6(-\frac{3}{2}) = 9 + 9 = 18$

$y = -7 + (-\frac{3}{2}) = -\frac{17}{2} = -8.5$

$z = 0$

$\boxed{(18, -8.5, 0)}$