13.3 Tools for Analyzing Vector Curves

In this section, we discuss three things: (1) Arc Length, (2) Unit Tangent/Normal Vectors, and (3) Curvature.

Arc Length

If a curve has the vector equation $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, then

$$\text{'Distance traveled from } t = a \text{ to } t = b\text{'} \;=\; \int_a^b |\mathbf{r}'(t)|\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

If the curve is *traversed once* from $t = a$ to $t = b$ (meaning the object doesn't turn around and/or do more than one full loop on the same path), then this distance gives the length of the curve. In which case, we call it the **arc length**.

For a given starting time a, we define

$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du \qquad \text{the arc length function.}$$

(this is the distance traveled by the object from time a to time t).

Arc Length Parametrization: Occasionally, we want to know the location in terms of the distance (instead of time). Consider the following questions:

1. "Where is an airplane located in 10 minutes?" This questions is asked in terms of time. So a parametric equation in terms of time, t, would be useful to answer the question.
2. "Where is an airplane located after traveling 20 miles?" This question is asked in terms of distance (arc length). So a parametric equation in terms of arc length, s, would be useful.

How to Reparametrize in Terms of Arc Length

1. Compute and simplify the arc length function from the given starting time.
2. Solve for t in terms of s.
3. Rewrite your curve in terms of s (by replacing t with what you found in the previous part).

Example: Reparameterize in terms of arc length for $x = 3t$, $y = 4\cos(t)$, $z = 4\sin(t)$.

Answer:
Step 1: Since $x'(t) = 3$, $y'(t) = -4\sin(t)$, $z'(t) = 4\cos(t)$, the arc length function is

$$s(t) = \int_0^t \sqrt{3^2 + 16\sin^2(t) + 16\cos^2(t)}\, dt = \int_0^t \sqrt{25}\, dt = 5t.$$

Step 2: Since $s = 5t$ we have $t = \frac{1}{5}s$.

Step 3: Replacing t by $\frac{1}{5}s$ gives $x = \frac{3}{5}s$, $y = 4\cos\left(\frac{1}{5}s\right)$, $z = 4\sin\left(\frac{1}{5}s\right)$.

Notes on why this is valuable:

- If you want to know where the object is located after it has traveled a distance of $s = 10$, you would quickly do it now by replacing s with 10.
- If you wanted to travel on the same curve but with a speed of 50 mph, you would note that $s'(t) = 50$, so integrating gives $s(t) = 50t$. Then you could replace s by $50t$ and you would have a parameterization that is traveling 50 mph.
- If you wanted to travel on this curve with a constant acceleration of 10 m/s^2 and start from rest, then $s''(t) = 10$, so you can integrate twice to get $s(t) = 5t^2$. Then you can replace s with $5t^2$ and you would have a parameterization that is traveling at a constant acceleration of 10 m/s^2.

As you see, distance plays a key role when you want to accurately motion on a curve.

Unit Tangent and Unit Normal Vectors

We already are able to find a vector that is tangent to our curve by using the derivative.
That is, $\mathbf{r}'(t) =$ 'a tangent vector'.

Now we will make it a unit vector, then use it to find vectors that are perpendicular (normal) to the tangent. Here is a collection of facts about these vectors:

$$\mathbf{T}(t) = \frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t) \quad \text{unit tangent}$$

$$\mathbf{N}(t) = \frac{1}{|\mathbf{T}'(t)|}\mathbf{T}'(t) \quad \text{principal unit normal ('inward' pointing normal)}$$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \quad \text{binormal vector (orthogonal to the tangent and unit normal vectors)}$$

Example: Find $\mathbf{T}(t)$, $\mathbf{N}(t)$ and $\mathbf{B}(t)$ for $x = 3t$, $y = 4\cos(t)$, $z = 4\sin(t)$.

Answer:
First, $\mathbf{r}'(t) = \langle 3, -4\sin(t), 4\cos(t)\rangle$. Its magnitude is $|\mathbf{r}'(t)| = \sqrt{9 + 16\sin^2(t) + 16\cos^2(t)} = \sqrt{25} = 5$. Therefore,

$$\mathbf{T}(t) = \frac{1}{5}\mathbf{r}'(t) = \langle \frac{3}{5}, -\frac{4}{5}\sin(t), \frac{4}{5}\cos(t)\rangle.$$

Now we find $\mathbf{T}'(t) = \langle 0, -\frac{4}{5}\cos(t), -\frac{4}{5}\sin(t)\rangle$. Its magnitude is $|\mathbf{T}'(t)| = \sqrt{0 + \frac{16}{25}\cos^2(t) + \frac{16}{25}\sin^2(t)} = \sqrt{\frac{16}{25}} = \frac{4}{5}$. Thus,

$$\mathbf{N}(t) = \frac{1}{4/5}\mathbf{T}'(t) = \langle 0, -\cos(t), -\sin(t)\rangle.$$

You won't be asked to compute the Binormal in this class, but I wanted to mention it here. You would find it by

$$\mathbf{B}(t) = \mathbf{T} \times \mathbf{N} = \langle \frac{4}{5}\sin^2(t) + \frac{4}{5}\cos^2(t), \frac{3}{5}\sin(t), -\frac{3}{5}\cos(t)\rangle = \langle \frac{4}{5}, \frac{3}{5}\sin(t), -\frac{3}{5}\cos(t)\rangle$$

Curvature

The curvature measures how quickly the direction of the unit tangent vector is changing with respect to distance. To make sure that the length of the tangent vector does not effect our computation, we start with the unit tangent vector $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$. Thus, the **curvature** of a curve is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \text{'magnitude of the rate of change of the unit tangent vector with respect to arc length'}$$

There are several equivalent ways to write this. Here are some variants:

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \qquad \text{(for 3 dimensions)}$$

$$\kappa(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}} \qquad \text{(for 2 dimensions)}$$

Example: Find the curvature of $x = 3t$, $y = 4\cos(t)$, $z = 4\sin(t)$.

Answer:
First, $\mathbf{r}'(t) = \langle 3, -4\sin(t), 4\cos(t)\rangle$ and $\mathbf{r}''(t) = \langle 0, -4\cos(t), -4\sin(t)\rangle$.

The speed is $|\mathbf{r}'(t)| = \sqrt{9 + 16\sin^2(t) + 16\cos(t)} = \sqrt{25} = 5$.

And $\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle 16\sin^2(t) + 16\cos^2(t), 12\sin(t), -12\cos(t)\rangle = \langle 16, 12\sin(t), -12\cos(t)\rangle$. Thus, $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{16^2 + 144\sin^2(t) + 144\cos^2(t)} = \sqrt{256 + 144} = \sqrt{400} = 20$.

Therefore, the curvature is

$$\kappa(t) = \frac{20}{5^3} = \frac{4}{25}.$$

Aside: The *radius of curvature* is the radius of the circular arc that best fits the curve at that point. It is equal to the reciprocal of the curvature. So in the last example, the radius of curvature is a constant $\frac{25}{4}$.

Example: Find the curvature of the 2D curve $f(x) = 5x^2$ and determine where curvature is maximum, also find $\lim\limits_{x\to\infty} \kappa(x)$.

Answer:
First, $f'(x) = 10x$ and $f''(x) = 10$. Thus

$$\kappa(x) = \frac{|10|}{(1 + (10x)^2)^{3/2}} = \frac{10}{(1 + 100x^2)^{3/2}}.$$

This function will be maximum when the denominator is as small as possible, which happens when $x = 0$. Another way to find this to take the derivative and set it equal to zero. Either way, you see that curvature is maximum when $x = 0$. Note that $\lim_{x\to\infty} \kappa(x) = 0$.