Math 126 Basic Summary of Facts

Vector Basics.

$\lvert\mathbf{v}\rvert = \sqrt{v_1^2 + v_2^2 + v_3^2}$	$c\mathbf{v} = \langle cv_1, cv_2, cv_3\rangle$	$\frac{1}{\lvert\mathbf{v}\rvert}\mathbf{v} =$ ‘unit vector in direction of $\mathbf{v}$
$\mathbf{u}+\mathbf{v} = \langle u_1+v_1, u_2+v_2, u_3+v_3\rangle$	$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$	$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
$\mathbf{u}\cdot\mathbf{v} = \lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert\cos(\theta)$	$\mathbf{u}\cdot\mathbf{v} = 0$ means orthogonal	θ is the angle if drawn tail to tail
$\mathbf{u}\times\mathbf{v} = \lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert\sin(\theta)$	$\mathbf{u}\times\mathbf{v}$ is orthogonal to both	$\lvert\mathbf{u}\times\mathbf{v}\rvert =$ parallelogram area
$\text{comp}_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert}$	$\mathbf{proj}_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert^2}\mathbf{a}$	

Comments: Know how to check/find vectors that are parallel or orthogonal. Be comfortable with computation, interpretations, and consequences.

Basic Lines, Planes and Surfaces (assume the constants a, b and c are positive in the last three rows):

Lines: $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$	$(x_0, y_0, z_0) =$ a point on the line $\langle a, b, c\rangle =$ a direction vector
Planes: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$	$(x_0, y_0, z_0) =$ a point on the plane $\langle a, b, c\rangle =$ a normal vector
Cylinder: One variable ‘missing’	Know basics of traces
Elliptical Paraboloid: $z = ax^2 + by^2$	Hyperboloid Paraboloid: $z = ax^2 - by^2$
Ellipsoid: $ax^2 + by^2 + cz^2 = 1$	Cone: $z^2 = ax^2 + by^2$
Hyperboloid of One Sheet: $ax^2 + by^2 - cz^2 = 1$	Hyperboloid of Two Sheets: $ax^2 + by^2 - cz^2 = -1$

Comments: You should be very good at finding lines/planes and naming basic shapes.

Basic Parametric and Polar in $\mathbb{R}^2$:

$\langle \frac{dx}{dt}, \frac{dy}{dt}\rangle =$ a tangent vector	$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$	$\frac{d^2y}{dx^2} = \frac{d/dt(dy/dx)}{dx/dt}$
$x = r\cos(\theta)$	$y = r\sin(\theta)$	$\tan(\theta) = \frac{y}{x}$
$x^2 + y^2 = r^2$	$\frac{dy}{dx} = \frac{(dr/d\theta)\sin(\theta) + r\cos(\theta)}{(dr/d\theta)\cos(\theta) - r\sin(\theta)}$	

Basic Parametric in $\mathbb{R}^3$:

$\mathbf{r}'(t) = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\rangle$	$\mathbf{r}''(t) = \langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\rangle$
$\int \mathbf{r}(t)\,dt = \left\langle \int x(t)dt, \int y(t)dt, \int z(t)dt \right\rangle$	Note: There are three constants of integration.
Arc Length $= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$	$s(t) = \int_0^t \sqrt{(x'(u))^2 + (y'(u))^2 + (z'(u))^2}\,du$
$\kappa(t) = \frac{\lvert\mathbf{r}'(t)\times\mathbf{r}''(t)\rvert}{\lvert\mathbf{r}'(t)\rvert^3}$	$\kappa(x) = \frac{\lvert f''(x)\rvert}{(1+f'(x)^2)^{3/2}} =$ ‘2D curvature’
$\mathbf{r}'(t) = \mathbf{v}(t) =$ velocity vector	$\lvert\mathbf{r}'(t)\rvert = \lvert\mathbf{v}(t)\rvert =$ speed
$\mathbf{r}''(t) = \mathbf{a}(t) =$ acceleration	$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt$ and $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt$
$\mathbf{T}(t) = \frac{1}{\lvert\mathbf{r}'(t)\rvert}\mathbf{r}'(t) =$ unit tangent	$\mathbf{N}(t) = \frac{1}{\lvert\mathbf{T}'(t)\rvert}\mathbf{T}'(t) =$ principal unit normal
$\mathbf{B}(t) = \mathbf{T}\times\mathbf{N} =$ binormal	$\mathbf{r}'(t)\times\mathbf{r}''(t) =$ a vector parallel to $\mathbf{B}$
$a_T = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\lvert\mathbf{r}'(t)\rvert}$	$a_N = \frac{\lvert\mathbf{r}'(t)\times\mathbf{r}''(t)\rvert}{\lvert\mathbf{r}'(t)\rvert}$

3D Parametric Comments and other notes:

- To find a normal plane: In the equation for the plane use $\mathbf{r}'(t)$ as the normal vector.
- To find an osculating plane: In the equation for the plane use any vector in the direction of $\mathbf{B}(t)$. The fastest way to do this is to find $\mathbf{r}'(t)\times\mathbf{r}''(t)$, this is a consequence of the fact that velocity and acceleration ($\mathbf{r}'(t)$ and $\mathbf{r}''(t)$) determine the same plane as $\mathbf{T}$ and $\mathbf{N}$ (i.e. all four of these vectors are in the same plane). So this gives us a slightly faster way to compute $\mathbf{B}(t)$.

Slopes on Surfaces.

Be able to find and graph the domain	Know the basics on level curves/contour maps
$f_x(x,y) = \frac{\partial z}{\partial x}$ = slope in x-direction	$f_y(x,y) = \frac{\partial z}{\partial y}$ = slope in y-direction
$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$	Tangent plane/linearization/total differential.
$f_{xx}(x,y) = \frac{\partial^2 z}{\partial x^2}$ = concavity in x-direction	$f_{yy}(x,y) = \frac{\partial^2 z}{\partial y^2}$ = concavity in y-direction
$f_{xy}(x,y) = \frac{\partial^2 z}{\partial y \partial x}$ = mixed second partial	$f_{xy}(x,y) = f_{yx}(x,y)$ (Clairaut's Theorem)
$D = f_{xx} \cdot f_{yy} - (f_{xy})^2$ = measure of concavity	$D < 0$ means concavity changes (saddle)
$D > 0$, $f_{xx} > 0$ means concave up all directions	$D > 0$, $f_{xx} < 0$ means concave down all directions

Comments:

- To find critical points: Find f_x and f_y, set them BOTH equal to zero, then COMBINE the equations and solve for x and y.
- To classify critical points: Find f_{xx}, f_{yy}, and f_{xy}. At the each critical point compute f_{xx}, f_{yy}, f_{xy} and D and make appropriate conclusions from the second derivative test.
- To find absolute max/min on a region: Find critical points inside the region. Then, over each boundary, substitution the xy-equation for the boundary into the surface to get a one variable function. Find the absolue max/min of the one variable function over each boundary. In the end, evaluate $f(x,y)$ at all the critical points inside the region and the critical numbers on the boundary to find the largest and smallest output.

Volumes under surfaces:

$$\iint_D f(x,y)\,dA = \text{signed volume 'above' the } xy\text{-axis, 'below' } f(x,y) \text{ and inside the region } D.$$

We also saw that $\iint_D 1\,dA = \text{area of } D$.

To set up a double integral: (1) Solving for integrand(s) (*i.e.* get $z = f(x,y)$). (2) Draw given xy-equations in the xy-plane. (label intersection points) (3) Draw any xy-equations that occur from intersection ($z = f(x,y)$ with $z = 0$ or the intersection of two given surfaces). (4) Set up the double integral(s) using the region for D.

Options for set up (you should also be able to take an integral that is already set up, draw the region, and reverse/change the order of integration):

$\iint_D f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dydx$,	$y = g(x)$ = bottom,	$y = h(x)$ = top
$\iint_D f(x,y)\,dA = \int_c^d \int_{p(y)}^{q(y)} f(x,y)\,dxdy$,	$x = p(y)$ = left,	$x = q(y)$ = right
$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{w(\theta)}^{v(\theta)} f(r\cos(\theta), r\sin(\theta))\,rdrd\theta$,	$r = w(\theta)$ = inner,	$r = v(\theta)$ = outer

We saw the following application: If $\rho(x,y)$ = density of a plate covering the region D, then

$$M = \text{total mass} = \iint_D \rho(x,y)\;dA \quad , \quad \overline{x} = \frac{\iint_D x\rho(x,y)\;dA}{\iint_D \rho(x,y)\;dA} \quad \text{and} \quad \overline{y} = \frac{\iint_D y\rho(x,y)\;dA}{\iint_D \rho(x,y)\;dA}$$

Taylor polynomials

$$T_1(x) = \sum_{k=0}^{1} \frac{f^{(k)}(b)}{k!}(x-b)^k \quad = f(b) + f'(b)(x-b).$$

$$T_2(x) = \sum_{k=0}^{2} \frac{f^{(k)}(b)}{k!}(x-b)^k \quad = f(b) + f'(b)(x-b) + \frac{f''(b)}{2!}(x-b)^2.$$

$$T_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(b)}{k!}(x-b)^k \quad = f(b) + f'(b)(x-b) + \frac{f''(b)}{2!}(x-b)^2 + \frac{f'''(b)}{3!}(x-b)^3.$$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(x-b)^k \quad = f(b) + f'(b)(x-b) + \frac{f''(b)}{2!}(x-b)^2 + \cdots + \frac{f^{(n)}(b)}{n!}(x-b)^n.$$

Taylor inequalities

$$\text{ERROR} = |f(x) - T_1(x)| \le \frac{M}{2!}|x-b|^2 \quad \text{, where } |f''(x)| \le M \text{ on the interval, and in general,}$$

$$\text{ERROR} = |f(x) - T_n(x)| \le \frac{M}{(n+1)!}|x-b|^{n+1} \quad \text{, where } |f^{(n+1)}(x)| \le M \text{ on the interval.}$$

Three types of error questions:

Given an interval $[b-a, b+a]$, find the $T_n(x)$ error bound:

1. Find $|f^{(n+1)}(x)|$.
2. Determine a bound (the maximum value if possible) for $|f^{(n+1)}(x)| \le M$ on the interval.
3. In Taylor's inequality $\frac{M}{(n+1)!}|x-b|^{n+1}$ replace M and replace x by an endpoint.

Find an interval so that $T_n(x)$ has a desired error:

1. Write $[b-a, b+a]$ and you will solve for a.
2. Find $|f^{(n+1)}(x)|$.
3. Determine a bound (the maximum value if possible) for $|f^{(n+1)}(x)| \le M$ on the interval, this will involve the symbol a.
4. In Taylor's inequality $\frac{M}{(n+1)!}|x-b|^{n+1}$ replace M and replace x by an endpoint (this will involve the symbol a).'
5. Then solve for a to get the desired error.

Given an interval $[b-a, b+a]$, find n so that $T_n(x)$ gives a desired error:
(There is no good general way to solve for the answer in this case, you just use trial and error).

1. Find the error for $n = 1$, then $n = 2$, then $n = 3$, etc. Once you get an error less than the desired error, you stop.
2. If you spot a pattern in the errors, then use the pattern to solve for the first time the error will be less than the desired error.

Taylor series

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k \quad = 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \cdots \quad \text{, for all } x.$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \quad = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 + \cdots \quad \text{, for all } x.$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \quad = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 + \cdots \quad \text{, for all } x.$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad = 1 + x + x^2 + x^3 + \cdots \quad \text{, for } -1 < x < 1.$$

Substituting into series (examples):

$$e^{2x^3} = \sum_{k=0}^{\infty} \frac{1}{k!} 2^k x^{3k} \quad = 1 + 2x^3 + \tfrac{2^2}{2!}x^6 + \tfrac{2^3}{3!}x^9 + \cdots \quad \text{, for all } x.$$

$$\sin(5x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} 5^{2k+1} x^{2k+1} \quad = 5x - \tfrac{5^3}{3!}x^3 + \tfrac{5^5}{5!}x^5 + \cdots \quad \text{, for all } x.$$

$$\cos(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{4k} \quad = 1 - \tfrac{1}{2!}x^4 + \tfrac{1}{4!}x^8 + \cdots \quad \text{, for all } x.$$

$$\frac{1}{1+3x} = \sum_{k=0}^{\infty} (-3)^k x^k \quad = 1 - 3x + 3^2x^2 - 3^3x^3 + \cdots \quad \text{, for } -1 < -3x < 1.$$

Multiplying out (examples):

$$x^3 e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^{k+3} \quad = x^3 + x^4 + \tfrac{1}{2!}x^5 + \tfrac{1}{3!}x^6 + \cdots \quad \text{, for all } x.$$

$$\frac{x^2}{1+2x} = \sum_{k=0}^{\infty} (-2)^k x^{k+2} \quad = x^2 - 2x^3 + 2^2x^4 - 2^3x^5 + \cdots \quad \text{, for } -1 < 2x < 1.$$

Integrating/Differentiating (examples):

$$-\ln(1-x) = \int_0^x \frac{1}{1-t}\,dt = \sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1} \quad = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \cdots \quad \text{, for } -1 < x < 1.$$

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+t^2}\,dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \quad = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \cdots \quad \text{, for } -1 < x < 1.$$

$$\int e^{x^3}\,dx = C + \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{3k+1} x^{3k+1} \quad = C + x + \tfrac{1}{2!(4)}x^4 + \tfrac{1}{3!(7)}x^7 + \cdots \quad \text{, for all } x.$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{k=0}^{\infty} k x^{k-1} \quad = 1 + 2x + 3x^2 + 4x^3 + \cdots \quad \text{, for } -1 < x < 1$$