Practice Finding Planes and Lines in $\mathbf{R}^3$

Here are several main types of problems you find in 12.5 and old exams pertaining to finding lines and planes:

LINES

1. Find an equation for the line that goes through the two points $A(1, 0, -2)$ and $B(4, -2, 3)$.
2. Find an equation for the line that is parallel to the line $x = 3 - t$, $y = 6t$, $z = 7t + 2$ and goes through the point $P(0, 1, 2)$.
3. Find an equation for the line that is orthogonal to the plane $3x - y + 2z = 10$ and goes through the point $P(1, 4, -2)$.
4. Find an equation for the line of intersection of the plane $5x + y + z = 4$ and $10x + y - z = 6$.

PLANES

1. Find the equation of the plane that goes through the three points $A(0, 3, 4)$, $B(1, 2, 0)$, and $C(-1, 6, 4)$.
2. Find the equation of the plane that is orthogonal to the line $x = 4 + t$, $y = 1 - 2t$, $z = 8t$ and goes through the point $P(3, 2, 1)$.
3. Find the equation of the plane that is parallel to the plane $5x - 3y + 2z = 6$ and goes through the point $P(4, -1, 2)$.
4. Find the equation of the plane that contains the intersecting lines $x = 4 + t_1$, $y = 2t_1$, $z = 1 - 3t_1$ and $x = 4 - 3t_2$, $y = 3t_2$, $z = 1 + 2t_2$.
5. Find the equation of the plane that is orthogonal to the plane $3x + 2y - z = 4$ and goes through the points $P(1, 2, 4)$ and $Q(-1, 3, 2)$.

LINES/PLANES/SPHERES AND INTERSECTIONS:

1. Find the intersection of the line $x = 3t$, $y = 1 + 2t$, $z = 2 - t$ and the plane $2x + 3y - z = 4$.
2. Find the intersection of the two lines $x = 1 + 2t_1$, $y = 3t_1$, $z = 5t_1$ and $x = 6 - t_2$, $y = 2 + 4t_2$, $z = 3 + 7t_2$ (or explain why they don't intersect).
3. Find the intersection of the line $x = 2t$, $y = 3t$, $z = -2t$ and the sphere $x^2 + y^2 + z^2 = 16$.
4. Find the intersection of the plane $3y + z = 0$ and the sphere $x^2 + y^2 + z^2 = 4$.

LINES (Solutions)

1. (a) *A position vector:* $\mathbf{r}_0 = \langle 1, 0, -2\rangle$
   (b) *A direction vector:* $\mathbf{v} = \langle 4-1, -2-0, 3-(-2)\rangle = \langle 3, -2, 5\rangle$
   (c) *Equation:* $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ which gives $x = 1+3t$, $y = 0-2t$, $z = -2+5t$.

2. (a) *A position vector:* $\mathbf{r}_0 = \langle 0, 1, 2\rangle$
   (b) *A direction vector:* $\mathbf{v} = \langle -1, 6, 7\rangle$ (Parallel to the other line, so we can use the same direction vector).
   (c) *Equation:* $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ which gives $x = 0-t$, $y = 1+6t$, $z = 2+7t$.

3. (a) *A position vector:* $\mathbf{r}_0 = \langle 1, 4, -2\rangle$
   (b) *A direction vector:* $\mathbf{v} = \langle 3, -1, 2\rangle$ (Orthogonal to the plane, so we can use the normal from the plane).
   (c) *Equation:* $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ which gives $x = 1+3t$, $y = 4-t$, $z = -2+2t$.

4. *Solution Method 1*: Find two points of intersection. There are many point we just need to find two.

   (a) First let's combine and simplify. Adding the equations gives $15x + 2y = 10$
   (b) Pick some numbers.
   - If $x = 0$, then we get $2y = 10$, so $y = 5$. And going back to the original equations and plugging in (to either one) we get $0 + 5 + z = 4$, so $z = -1$. Hence, $(0, 5, -1)$ is a point on the line we desire.
   - If $y = 0$, then we get $15x = 10$, so $x = 2/3$. And going back to the original equation we get $5(2/3) + 0 + z = 4$, so $z = 4 - 10/3 = 2/3$. Thus another point is $(2/3, 0, 2/3)$.

   You can check that these points work in both equations. Now we can use the standard line method.

   (c) *A position vector:* $\mathbf{r}_0 = \langle 0, 5, -1\rangle$
   (d) *A direction vector:* $\mathbf{v} = \langle 2/3-0, 0-5, 2/3-(-1)\rangle = \langle 2/3, -5, 5/3\rangle$.
   (e) *Equation:* $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ which gives $x = 0 + 2/3t$, $y = 5-5t$, $z = -1+5/3t$.

   Solution Method 2: Find one point of intersection then use the cross-produce of the normal for the direction.

   (a) For this method you still have to find one point of intersection. So for example $(0, 5, -1)$ as we did above.
   (b) The cross product of the normals for each plane will give a vector that is parallel to the line (picture it). So this is another way to get a direction vector. That would give $\langle 5, 1, 1\rangle \times \langle 10, 1, -1\rangle = \langle -1-1, -(-5-10), 5-10\rangle = \langle -2, 15, -5\rangle$.
   (c) *A position vector:* $\mathbf{r}_0 = \langle 0, 5, -1\rangle$
   (d) *A direction vector:* $\mathbf{v} = \langle -2, 15, -5\rangle$. (Note this is parallel to the direction vector we got with method 1).
   (e) *Equation:* $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ which gives $x = 0-2t$, $y = 5+15t$, $z = -1-5t$. Remember you and your classmate may have different parameterizations and both be correct. But your direction vectors should be parallel.

PLANES (Solutions)

1. (a) *A position vector:* $\mathbf{r}_0 = \langle 0, 3, 4\rangle$

   (b) *A normal vector:* $\mathbf{AB} = \langle 1, -1, -4\rangle$ and $\mathbf{AC} = \langle -1, 3, 0\rangle$, so one normal vector is $\mathbf{n} = \langle 1, -1, -4\rangle \times \langle -1, 3, 0\rangle = \langle 12, 4, 2\rangle$

   (c) *Equation:* $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$ which gives $12(x-0) + 4(y-3) + 2(z-4) = 0$, or more simply $12x + 4y + 2z - 20 = 0$.

2. (a) *A position vector:* $\mathbf{r}_0 = \langle 3, 2, 1\rangle$

   (b) *A normal vector:* $\mathbf{n} = \langle 1, -2, 8\rangle$ (Orthogonal to the line, so the direction vector for the line is a normal to the plane).

   (c) *Equation:* $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$ which gives $(x-3) - 2(y-2) + 8(z-1) = 0$, or more simply $x - 2y + 8z - 7 = 0$.

3. (a) *A position vector:* $\mathbf{r}_0 = \langle 4, -1, 2\rangle$

   (b) *A normal vector:* $\mathbf{n} = \langle 5, -3, 2\rangle$ (Parallel to the other plane, so same normal works).

   (c) *Equation:* $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$ which gives $5(x-4) - 3(y+1) + 2(z-2) = 0$, or more simply $5x - 3y + 2z - 27 = 0$.

4. Note that the lines intersect at $t_1 = 0$ and $t_2 = 0$, which gives the point $P(4, 0, 1)$. We can quickly find three points by also plugging in $t_1 = 1$ and $t_2 = 1$ which gives $Q(5, 2, -2)$ and $R(1, 3, 3)$. So we have three points. Note also that $\mathbf{PQ} = \langle 1, 2, -3\rangle$ and $\mathbf{PR} = \langle -3, 3, 2\rangle$ (so I really didn't have to find Q and R I could have just grabbed the direction vectors from the lines).

   (a) *A position vector:* $\mathbf{r}_0 = \langle 4, 0, 1\rangle$

   (b) *A normal vector:* $\mathbf{n} = \langle 1, 2, -3\rangle \times \langle -3, 3, 2\rangle = \langle 13, 7, 9\rangle$.

   (c) *Equation:* $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$ which gives $13(x-4) + 7(y+0) + 9(z-1) = 0$, or more simply $13x + 7y + 9z - 61 = 0$.

5. You have two vectors parallel to the plane. One is $\mathbf{PQ} = \langle -2, 1, -2\rangle$ and the other is the normal from the given plane which is $\langle 3, 2, -1\rangle$.

   (a) *A position vector:* $\mathbf{r}_0 = \langle 1, 2, 4\rangle$

   (b) *A normal vector:* $\mathbf{n} = \langle -2, 1, -2\rangle \times \langle 3, 2, -1\rangle = \langle 3, -8, -7\rangle$.

   (c) *Equation:* $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$ which gives $3(x-1) - 8(y-2) - 7(z-4) = 0$, or more simply $3x - 8y - 7z + 41 = 0$.

LINES/PLANES/SPHERES AND INTERSECTIONS (Solutions):

1. (a) *Combine and find t:* $2(3t) + 3(1 + 2t) - (2 - t) = 4$ gives $6t + 3 + 6t - 2 + t = 4$, so $13t = 3$ and $t = 3/13$.

   (b) *Get the point:* Thus, $x = 9/13$, $y = 1 + 6/13 = 29/13$, and $z = 2 - 3/13 = 23/13$.

2. (a) *Combine and find t_1 and t_2:*

      i. $1 + 2t_1 = 6 - t_2$ implies that $t_2 = 5 - 2t_1$.

      ii. $3t_1 = 2 + 4t_2$ combined with the fact just obtained gives $3t_1 = 2 + 4(5 - 2t_1)$ which gives $3t_1 = 22 - 8t_1$, so $11t_1 = 22$
      Hence, $t_1 = 2$ and going back, we also get $t_2 = 1$. Thus, the only parameters that simultaneously work to equate x and y are $t_1 = 2$ and $t_2 = 1$. Now we check the third equation.

      iii. $5t_1 = 3 + 7t_2$. Plugging in $t_1 = 2$ and $t_2 = 1$ we get $10 = 3 + 7$, it works!

   (b) *Get the point:* Thus, $x = 5$, $y = 6$, and $z = 10$ is the point where the two lines intersect.

3. (a) *Combine and find t:* $(2t)^2 + (3t)^2 + (-2t)^2 = 16$ gives $4t^2 + 9t^2 + 4t^2 = 16$, so $17t^2 = 16$ and $t = \pm\sqrt{16/17} = \pm 4/\sqrt{17}$.

   (b) *Get the points:* Thus, the two points of intersection are $(8/\sqrt{17}, 12/\sqrt{17}, -8/\sqrt{17})$ and $(-8/\sqrt{17}, -12/\sqrt{17}, 8/\sqrt{17})$.

4. (a) *Combine* Since $z = -3y$ we get $x^2 + y^2 + (-3y)^2 = 4$ which gives $x^2 + 10y^2 = 4$.

   (b) *What is this:* So every point that satisfies $x^2+10y^2 = 4$ with $z = -3y$ is a point of intersection. That is really the best we can do. (In terms of looking from above, meaning the projection onto the xy-plane, $x^2 + 10y^2 = 4$ would look like an ellipse. Also, $z = -3y$ is a plane through the origin and if you visualize the intersection you will see that it is just a great circle of the sphere). In any case, the point is that the intersection of two surfaces is typically a curve in two dimensions, not just a point.